
Introductory Mathematics for Economics

ECO104



University of Ibadan Distance Learning Centre
Ibadan Open and Distance Learning Course Series Development

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Vice-Chancellor's Message

The Distance Learning Centre is building on a solid tradition of over two decades of service in the provision of External Studies Programme and now Distance Learning Education in Nigeria and beyond. The Distance Learning mode to which we are committed is providing access to many deserving Nigerians in having access to higher education especially those who by the nature of their engagement do not have the luxury of full time education. Recently, it is contributing in no small measure to providing places for teeming Nigerian youths who for one reason or the other could not get admission into the conventional universities.

These course materials have been written by writers specially trained in ODL course delivery. The writers have made great efforts to provide up to date information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly.

In addition to provision of course materials in print and e-format, a lot of Information Technology input has also gone into the deployment of course materials. Most of them can be downloaded from the DLC website and are available in audio format which you can also download into your mobile phones, IPod, MP3 among other devices to allow you listen to the audio study sessions. Some of the study session materials have been scripted and are being broadcast on the university's Diamond Radio FM 101.1, while others have been delivered and captured in audio-visual format in a classroom environment for use by our students. Detailed information on availability and access is available on the website. We will continue in our efforts to provide and review course materials for our courses.

However, for you to take advantage of these formats, you will need to improve on your I.T. skills and develop requisite distance learning Culture. It is well known that, for efficient and effective provision of Distance learning education, availability of appropriate and relevant course materials is a *sine qua non*. So also, is the availability of multiple plat form for the convenience of our students. It is in fulfilment of this, that series of course materials are being written to enable our students study at their own pace and convenience.

It is our hope that you will put these course materials to the best use.



Prof. Abel Idowu Olayinka

Vice-Chancellor

Foreword

As part of its vision of providing education for “Liberty and Development” for Nigerians and the International Community, the University of Ibadan, Distance Learning Centre has recently embarked on a vigorous repositioning agenda which aimed at embracing a holistic and all encompassing approach to the delivery of its Open Distance Learning (ODL) programmes. Thus we are committed to global best practices in distance learning provision. Apart from providing an efficient administrative and academic support for our students, we are committed to providing educational resource materials for the use of our students. We are convinced that, without an up-to-date, learner-friendly and distance learning compliant course materials, there cannot be any basis to lay claim to being a provider of distance learning education. Indeed, availability of appropriate course materials in multiple formats is the hub of any distance learning provision worldwide.

In view of the above, we are vigorously pursuing as a matter of priority, the provision of credible, learner-friendly and interactive course materials for all our courses. We commissioned the authoring of, and review of course materials to teams of experts and their outputs were subjected to rigorous peer review to ensure standard. The approach not only emphasizes cognitive knowledge, but also skills and humane values which are at the core of education, even in an ICT age.

The development of the materials which is on-going also had input from experienced editors and illustrators who have ensured that they are accurate, current and learner-friendly. They are specially written with distance learners in mind. This is very important because, distance learning involves non-residential students who can often feel isolated from the community of learners.

It is important to note that, for a distance learner to excel there is the need to source and read relevant materials apart from this course material. Therefore, adequate supplementary reading materials as well as other information sources are suggested in the course materials.

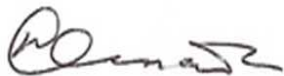
Apart from the responsibility for you to read this course material with others, you are also advised to seek assistance from your course facilitators especially academic advisors during your study even before the interactive session which is by design for revision. Your academic advisors will assist you using convenient technology including Google Hang Out, You Tube, Talk Fusion, etc. but you have to take advantage of these. It is also going to be of immense advantage if you complete assignments as at when due so as to have necessary feedbacks as a guide.

The implication of the above is that, a distance learner has a responsibility to develop requisite distance learning culture which includes diligent and disciplined self-study, seeking available administrative and academic support and acquisition of basic information technology skills. This is why you are encouraged to develop your computer skills by availing yourself the opportunity of training that the Centre’s provide and put these into use.

In conclusion, it is envisaged that the course materials would also be useful for the regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks. We are therefore, delighted to present these titles to both our distance learning students and the university's regular students. We are confident that the materials will be an invaluable resource to all.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.

A handwritten signature in black ink, appearing to read 'Bayo Okunade', with a stylized flourish at the end.

Professor Bayo Okunade

Director

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About this course

Introductory Mathematics for Economics ECO104 has been produced by University of Ibadan Distance Learning Centre. All Economics Course Materials produced by University of Ibadan Distance Learning Centre are structured in the same way, as outlined below.

How this course is structured

The course overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you.
- What you will already need to know.
- What you can expect from the course.
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

- Study skills.
- Where to get help.
- Course assessments and assignments.
- Activity icons.
- Study sessions.

We strongly recommend that you read the overview *carefully* before starting your study.

The course content

The course is broken down into modules. Each module comprises:

- An introduction to the module content.
- Learning outcomes.
- New terminology also provided as glossary item in virtual classroom.
- Content of study sessions.
- A module summary.
- Assessments and/or assignments, as applicable.



Bibliography

The bibliography section points you in direction of additional resources at the end of this; these may be books, articles or websites.

Your comments

After completing this course, Introductory Mathematics for Economics, we would appreciate it if you would take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments on:

- Course content and structure.
- Course reading materials and resources.
- Course assessments.
- Course assessments.
- Course duration.
- Course support (assigned tutors, technical help, etc).
- Your general experience with the course provision as a distance learning student.

Your constructive feedback will help us to improve and enhance this course.



Course overview

Welcome to Introductory Mathematics for Economics ECO104

Mathematics is an integral part of economics and understanding basic concepts is vital. The foundations of economic theory are based on mathematical models. Thus, a thorough understanding of the economic content of such models is not possible without a clear understanding of the mathematical concepts that underpin the modelling. This clearly written manual will help you to develop quantitative skills up to the required level for a general Economics course.

After a review of the fundamentals of algebraic methods, sets, numbers, and functions, the manual covers limits, system of equations, matrix algebra, and calculus. To develop student's problem-solving skills, the manual works through a large number of examples and economic applications. The course also entails the presentation of research project, which is a critical element of the course and should be taken very seriously. Once you learn the basic mathematical methods, you can easily transition to applying the methods in real life economic situations.

Introductory Mathematics for Economics ECO104—is this course for you?

This course is intended for people who intends to: understand the mathematical methods that are most widely used in economics. It is *a compulsory course* which aims to provide the opportunity for students to be equipped with the skills of basic mathematical methods that have become the language of communication in modern Economics.

The only prerequisite is “high school further mathematics”.

Course outcomes

Upon a successful completion of Introductory Mathematics for Economics ECO104 you will be able to:



Outcomes

- *Know* how to read, understand, and construct mathematical proofs, and appreciate their role in the derivation of mathematical concepts and structures.
- *Apply* mathematical methods and techniques that are formulated in abstract settings to concrete economic applications.



Getting around this course manual

Margin icons

While working through this you will notice the frequent use of margin icons. These icons serve to “signpost” a particular piece of text, a new task or change in activity; they have been included to help you to find your way around this.

A complete icon set is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your study.

			
Activity	Assessment	Case study	Discussion
			
Help	Outcomes	Reading	Reflection
			
Study skills	Summary	Time	Tip



Study Session1

Use of Mathematics in Economics

Introduction

In Economics, mathematics is basically a tool. It is a tool that serves as an approach to model human behaviour. This study session looks at how mathematics is employed in economic situation. It also compares mathematical economics with nonmathematical economics on one hand, and mathematical economics with econometrics on the other hand.

Upon completion of this study session you will be able to:



Learning Outcomes

- *recognise* the use of mathematics in economics.
- *compare* mathematical and nonmathematical economics.
- *Identify* the similarity and difference between mathematical economics and econometrics.

Terminology

Mathematical Economics:	an approach of using symbols and equations to explain economic behaviour.
Nonmathematical Economics:	an approach of using words to explain economic behaviour.
Econometrics:	a quantitative approach that is dominated by statistical methods to describe economic system.

Mathematical Models

The world is full of several economic activities such as consumption, production and exchange. These activities are too complex to be studied on a whole. These activities can only be considered as variables that influence one another. With these variables, mathematical models can be constructed to observe the real life activities. So, Mathematical models are abstractions that capture real life phenomenon. For economists, real life situations are observed over time, from which a theory is developed. Then a model is constructed based on predictions from the theory. Economists test these predictions and compare their results with actual phenomenon. The results of the comparisons then determines whether the model is accepted, rejected or modified to serve as an abstraction from the real world.



Mathematical Economics vs. Nonmathematical Economics

Economic analysis is a systematic approach of applying data to a theory to determine optimum use of scarce resources, involving comparison of two or more alternatives in achieving a specific objective under given assumptions and constraints.

Deductive reasoning

logical move from general premise to a more precise conclusion

Mathematical and non-mathematical economics are alternative approaches to **economic analysis**. Hence, there is no fundamental difference between the two. The main difference between them is that mathematical economics uses symbols and equations while nonmathematical economics uses words to explain economic behaviour. Both approaches are very useful in economic analysis. The mathematical approach is however, more convenient and conducive to use in **deductive reasoning**.

ITQ

Question

If a person explains economic issues verbally, such approach can be called _____.

Feedback

Non-mathematical economics

Observably, the mathematical approach has the following advantages:

- concise and precise use of language;
- speeds up the analytical process;
- keeps economists from unintentional adoption of unwanted implicit assumptions; and
- allows economists treat the general n-variable case.

Although the mathematical approach could be difficult to comprehend at first, the effort and time taken to understand it eventually pays off when explanation of economic behaviour becomes faster.

Mathematical Economics vs. Econometrics

It is possible to confuse mathematical economics with econometrics. Both mathematical economics and econometrics are *quantitative* branches of economics, but they are not the same. Mathematical economics refers to the application of mathematics to the purely *theoretical* aspects of economic analysis while econometrics is concerned with the study of *empirical* observations using statistical methods of estimation and hypothesis testing.

In mathematical economics, focus is on *theoretical* issues while econometrics deals with *empirical* issues.

As earlier noted, the focus of this course will be on mathematical economics. Consequently, we are interested in developing a theoretical framework for economic analysis which is based on mathematical formulations.



Study session summary



Summary

In this Study session you learned that in Economics we:

1). Observe real world phenomena =2). Develop a Theory=3). Formulate tests to check predictions =4). Compare predictions of the model versus actual phenomenon:

Then, we either

Accept

Reject

Modify

the model as an abstraction from the real world.

We went further to compare mathematical relationship with nonmathematical relationship. Lastly, we looked at the similarities and differences between mathematical relationship and econometrics.

Assessment



Assessment

SAQ 1.1 (tests Learning Outcome 1.1)

Identify three advantages and two disadvantages of using mathematics in economics.

SAQ 1.2 (tests Learning Outcome 1.2)

An economist wants to show the relationship between price and quantity supplied. Rather than showing it verbally, he decides to apply quantitative methods, what approach is best for him to adopt?

SAQ 1.3 (tests Learning Outcome 1.3)

What is the main difference between mathematical economics and econometrics?



Study Session2

Review of Algebraic Methods

Introduction

As earlier noted, mathematics is an integral part of economics and understanding basic concepts of mathematics is as such vital. Many adult learners come into economics courses without having studied mathematics for a number of years. This study session will help you to review fundamental principles in mathematics.

Upon completion of this study session you will be able to:



Learning Outcomes

- *use* mathematical models in relation to the real number system
- *collect* objects in sets.
- *determine* if a relation is a function, use function notation, and find the domain and range of a function

Terminology

Sets:	A well defined collection of objects.
Relations:	A set of input and output values, usually represented in ordered pairs.
Functions:	A special relationship between values.

The Real Number System

As discussed in Study session **2.1**, Mathematical models are made up of a combination of equations and variables. Each of these - equations and variables - consists of numbers. We will therefore discuss the numbers system. Numbers can be either real or imaginary. Real numbers include all rational and irrational numbers. Thus real numbers are integers, fractions and decimals. They even include all positive and negative numbers. While an imaginary number is one that when squared, gives a negative result. In this session, focus will be on **real number system**.

The real number system is a union of rational and irrational numbers.

Rational Numbers

Rational numbers are numbers that can be expressed as a ratio of 2 integers/numbers. Rational numbers are further subdivided into two: integers and fractions.



- a. Integers – integers are whole numbers which are expressed only as a fraction of 1. Integers can be either positive or negative. Positive integers are whole numbers which are positive e.g. 1, 2, 3... while negative integers are negative whole numbers e.g. -1, -2, -3, ... 0 is a unique integer because it is neither positive nor negative.

All integers are called the set of all integers.

- b. Fractions – fractions are numbers which are expressed as a fraction of other numbers. Fractions can also be positive e.g. $\frac{1}{2}$, $\frac{3}{4}$... or negative e.g. $-\frac{5}{7}$, $-\frac{7}{9}$...

All fractions are collectively called the set of all fractions.

Irrational Numbers

Irrational numbers are numbers that cannot be expressed as ratios of a pair of integers, e.g. $\pi = 3.1415$.

As you can see in Figure 2.1, when all numbers are placed on a ruler, positive integers would lie to the right of zero while negative integers will lie to the left of zero. The fractions would lie between the integers while the irrational numbers would fall between the rational numbers.

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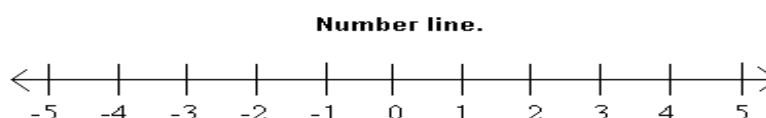


Figure 2.1

The whole group of these numbers (rational and irrational numbers) are called the set of all real numbers, denoted by \mathbb{R} .

Sets

A set is a collection of objects called *members* or *elements* of a set. Sets are represented by upper case letters while the elements are denoted by lower case letters, e.g. a set can be represented by $A = \{a, b, c, \dots\}$. a means a is an element of A . For any element x which is not a member of set A , you write $x \notin A$ meaning x is not in A . *

Representing Sets

Ordered set A set that is enclosed with parenthesis i.e. $()$.

Unordered set A set that is enclosed with braces i.e. $\{\}$.

There are two methods of representing sets.

- Enumeration or listing method:** In this method, a set is represented by writing out the elements one after the other, for example $A = [2, 4, 6, 8, 10, 12, 14]$
- Descriptive method:** Here, the elements in a set are narrated. For set A above, it will be expressed descriptively as: $A = [x: x \text{ is even numbers between 1 and 15}]$ or $A = [x: x \text{ is even numbers, } 1 \leq x \leq 15]$



Note that the ‘:’ and ‘/’ signs mean “such that”, that is, A is a set of x such that x is even and is between 5 and 30.

Relationships between Sets

Assuming there are two sets, X and Y , such that:

$$X = \{1, 2, 3, 5, 9, 16\} \text{ and } Y = \{2, 3, 4, 5, 7, 9, 14\}$$

These two sets can be related in two ways; their *intersection* and their *union*.

The intersection of sets X and Y is set $Z = \{2, 3, 5, 9\}$, that is, a set of numbers that are common to X and Y . The notation for intersection of X and Y is $Z = X \cap Y$. In descriptive form, it can be expressed as $Z = \{x:$

$$x \in X \text{ and } x \in Y\}$$

On the other hand, the union of sets X and Y is set $V =$

$\{1, 2, 3, 4, 5, 9, 14, 16\}$. This is denoted as $V = X \cup Y$. In simple form, this is

$$V = \{x/x \in X \text{ or } x \in Y\}$$

Another form of linkage among sets exists when a set is a subset of a bigger set. For example, if $A = \{3, 5, 8\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, we can say A is a subset of B since all elements in A are contained in B . In other words, B is a super set of A . A as a subset of B is denoted as $A \subset B$, while B as a superset of A is denoted as $A \subset B$.*

The relative difference between two sets say A and B is the set of all elements that are in the larger set and not in the smaller set.

ITQ

Question

If A is a set of all positive integers from 1 to 8 and B is a set of all odd numbers from 1 to 9, what will be their intersection?

Feedback

$A = [1, 2, 3, 4, 5, 6, 7, 8]$ and $B = [1, 3, 5, 7, 9]$, then their intersection will be

$$A \cap B = [1, 3, 5, 7]$$

Special Types of Sets

There are some special sets namely:

1. **Universal Set:** It is the set that contains all other subsets in a given situation. For example, if subset A = (all economics students in Idia hall), subset B = (all physics students in Idia hall), subset C = [all education students in Idia Hall), the

universal set can be $U =$ (all students in Idia hall) such that students in Idia hall that are not in economics, physics or education can form another subset. Universal sets are denoted

by U or



The set of all elements in the universal set but which are not in A is known as the complement of A denoted as A^c or A'

2. **Null or Empty Set:** This is any set that has no element. It is denoted as \emptyset . The intersect of any set and its complement will be a null set.
3. **Disjointed Sets:** Any two sets that have nothing in common are disjointed sets, for example, a set and its complement. The collection of all disjointed subsets of a universal set is the *partition* of the universal set.
4. **Power Set of a Set:** This is the set of all subsets that can be found in a set. The larger the number of elements in a set, the larger the power set of that set.
5. **Singleton:** This is a set that contains only one element. E.g. $X = \{a\}$
6. Sets of Natural numbers and fractions

Laws of Set Operations

Cummulative Law: This law states that for every two sets A and B, $[A \cup B] = [B \cup A]$ and $[A \cap B] = [B \cap A]$. This means that the arrangement of the sets does not matter.

Associative Law: This law states that for every three sets A, B, C, if $[A \cup B] \cup C = A \cup [B \cup C]$, that is, the sets merged first do not matter. Also $[A \cap B] \cap C = A \cap [B \cap C]$

Distributive Law: This law states that for every three sets, A, B and C.

$$[A \cup B] \cap C = [A \cap C] \cup [B \cap C]$$

$$[A \cap B] \cup C = [A \cup C] \cap [B \cup C]$$

De Morgan's Law: This law states that for every two sets A and B, the complement of their union is equal to the intersect of their independent complements. Also, the complement of A intersection B is equal to the union of their independent complements.

$$\text{e.g. } [A \cup B]^c = A^c \cap B^c ; [A \cap B]^c = A^c \cup B^c$$

ITQ

Question

If $A = \{2,4,6,8\}$, $B = \{1,3,5,7\}$ and $C = \{7,8,9\}$, test if the associative law holds for A, B and C.

Feedback

$$[A \cup B] \cup C = A \cup [B \cup C]$$

$$[A \cup B] = \{1,2,3,4,5,6,7,8\}, [A \cup B] \cup C = \{1,2,3,4,5,6,7,8,9\}$$



$$[B \cap C] = [1,3,5,7,8,9], A \cap [B \cap C] = [1,2,3,4,5,6,7,8,9]$$

Inclusion and Exclusion Principle

The principle states that in case of two sets A and B, the *cardinality* of set A which is the number of elements in A, that is, $n[A]$ and the cardinality of set B, $n[B]$ minus the cardinality of the intersection of A and B, $n[A \cap B]$ equals the union of A and B.

e.g. $n[A \cup B] = n[A] + n[B] - n[A \cap B]$

Conversely, the cardinality of $[A \cap B]$ is;

e.g. $n[A \cap B] = n[A] + n[B] - n[A \cup B]$

For a three-set case, if A, B and C are non-empty sets, the cardinality of their union is

$$n[A \cup B \cup C] = n[A] + n[B] + n[C] - n[A \cap B] - n[A \cap C] - n[B \cap C] + n[A \cap B \cap C].$$

This means the cardinality of the union of three sets, A, B and C is the sum of their individual coordinates minus the cardinalities of their pair wise intersection plus the cardinality of their intersection.

ITQ

Question

Let $A = [10,20,30,40,50]$, what will be the cardinality of A?

Feedback

The cardinality of A, $n[A] = 5$

Ordered and Unordered Sets

The elements in a set can be *ordered* or *unordered*. When the arrangement of elements in a set is significant, the set is *ordered* by enclosing it with parentheses i.e. (). For instance, we can provide two different ordered pairs denoted by (a,b) and (b,a), which implies that $(a,b) \neq (b,a)$ unless $a=b$. However, when the arrangement of elements in a set is not significant, the set is *unordered* by using the symbol of braces { }. E.g. {a,b} means the order in which elements a and b appear is not important.

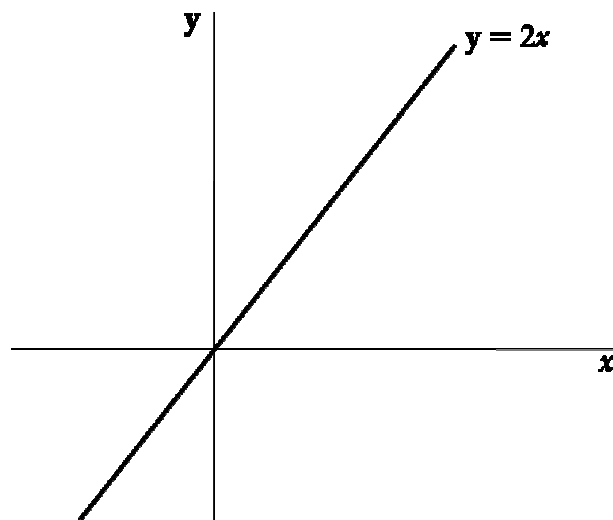
Relations and Functions

A **relation** is any collection of ordered pairs. For the ordered pair (x,y) where both x and y include all real numbers, the relation between x and y will be given by any subset of (x,y). E.g. the set { (x,y) | x < y }



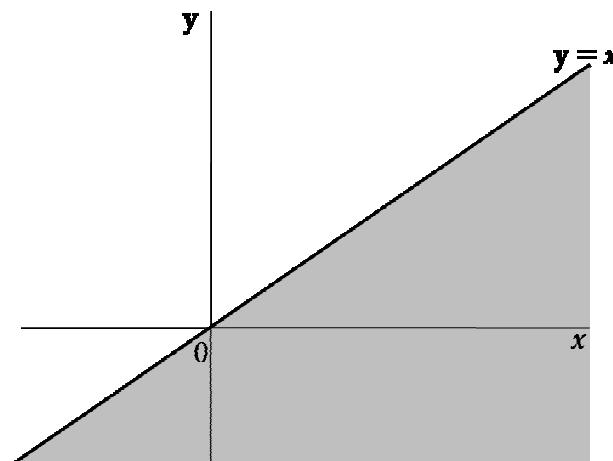
is a set of ordered pairs and the ordered pairs it includes are e.g. (1, 2), (0, 0), (-1, -2). As you can see in Fig 2.1, the set constitutes a relation which is graphically represented by the set of points lying on the straight line.

Figure 2.2



The set is also a set of ordered pairs such as (1, 0), (1, 1), and (1, -4). It constitutes a relation and is represented graphically by the set of all points in the shaded area which satisfy the inequality as you can see in Fig 2.2.

Figure 2.3



A unique value may not always be determinable in a given relation. For instance, in if , can take various values e.g. 0, 16, 4 and still satisfy the relation.

It is possible to have a relation where each value has only one corresponding value. E.g. . Here, is a **function** of , denoted by . Function is a set of ordered pairs with the property that any value uniquely determines a value. *A function must be a relation, but a relation may not be a function.*



A function implies a rule by which a transformation takes place. It is therefore, called mapping or transformation e.g. with $f: X \rightarrow Y$, it means a rule by which the set X is 'mapped' or 'transformed' into the set Y .

This is denoted by

where arrow indicates mapping and letter f specifies a rule of mapping

ITQ

Question

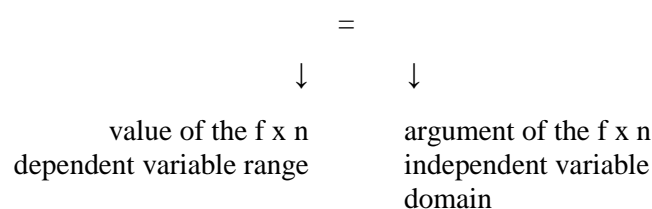
Which of the following is not a function?

- a.
- b.
- c.
- d.

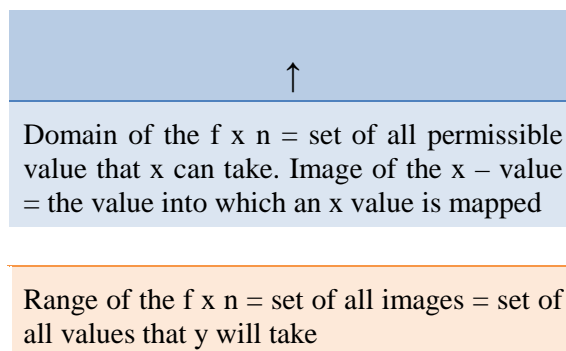
Feedback

The right option is b. Although, options a, c and d are relations, however, they are not functions because they do not have specific values for y

In the function $f: X \rightarrow Y$, x is referred to as the argument of the function, and y is called the value of the function. x is also called the independent variable and y the dependent variable. The set of all permissible values that x can take in a given context is known as the domain of the function, which may be a subset of the set of all real numbers. The y value into which an x value is mapped is called the image of that x value. The set of all images is called the range of the function, which is the set of all values that the y variable will take. Thus, the domain pertains to the independent variable x , and the range has to do with the dependent variable y as shown:



This is further exemplified below.



**Example 1**

The total cost C of a firm per day is a function of its daily output Q : $C = 200 + 7Q$. The firm has a capacity limit of 120 units of output per day. What are the domain and range of the cost function?

Solution:

Q can only vary between 0 and 120

Therefore, domain = set of values $0 \leq Q \leq 120$

$$= \{Q \mid 0 \leq Q \leq 120\}$$

min. C value is when $Q = 0 \rightarrow C = 200$

max. C value is when $Q = 120 \rightarrow C = 1040$

therefore, range = $\{C \mid 200 \leq C \leq 1040\}$

Example 2

If the domain of the function $y = 6 + 4x$ is the set $\{x \mid 1 \leq x \leq 5\}$, find the range of the function and express it as a set.

Solution

min. value is when

max. value is when

Therefore, range =

Example 3

For the function $y = 6 + 4x$, if the domain is the set of all non-negative real numbers, what will its range be?

Solution

min. value is when

max. y value is when

Therefore, range =

Study session summary

**Summary**

In this study session you learned how to use mathematical models in relation to the real number system, and also learned how to collect objects in sets. We capped this session by determining if a relation is a function, use function notation, and find the domain and range of a function



Assessment



Assessment

SAQ 2.1 (tests Learning Outcome 2.1)

What are the possible limitations of using irrational numbers in mathematical modelling?

SAQ 2.2 (tests Learning Outcome 2.2)

A school has 260 students. Of these total, 240 students offer at least one of the three social science subjects, Economics, Sociology and Geography. 70 offer Economics, 130, Sociology and 150 offer Geography. 50 students offer Economics and Sociology, 105 offer Sociology and Geography. If 40 students offer the three subjects, how many students offer Economics and Geography?

SAQ 2.3 (tests Learning Outcome 2.3)

A textile firm faces a cost function $C = 300 + 8Q$. The firm's maximum capacity daily is 150 units of textiles. What is the domain and range of the firm's cost function?



Study Session 3

Types of Functions

Introduction

Observing the function , this function informs that there is a rule by which the set x is mapped into the set y . However, we do not know the actual rule of mapping. By studying different types of functions, we can know the exact rule of mapping associated with different functions. You will therefore be exposed to different types of functions in this study session.

Upon completion of this study session you will be able to:



Learning Outcomes

- *construct* functions
- *identify* at least four types of functions.
- *graph* each of the identified types of functions.

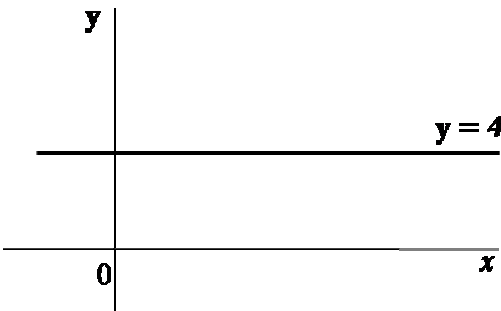
Terminology

Algebraic Functions:	Functions constructed using only a finite number of elementary operations together with the inverses of functions capable of being so constructed.
Transcendental Function:	A function not expressible as a finite combination of the algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root.

Constant Function

A constant function is a function that has a range that includes only one element. As you can see in Fig 3.1, the value of remains the same different values of .

Figure 3.1
Linear



With different values of x , the value of a constant function remains the same.



Polynomial Functions

A polynomial function is a function that consists of multiple terms which can be expressed in higher powers. E.g.

$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is a polynomial function that consists of only one independent variable x , with x expressed in higher powers up to the order n .

NOTE that

This shows that we can have different cases of polynomial functions depending on the power of the independent variable.

Note

- a. When the highest power of the independent variable = 0, we have a constant function as earlier discussed
i.e.
- b. When the highest power of the independent variable = 1, we have a linear function → first-degree polynomial
i.e.
- c. When the highest power of the independent variable = 2, we have a quadratic function → second-degree polynomial
i.e.

You can study the two graphs in Fig 3.2 below

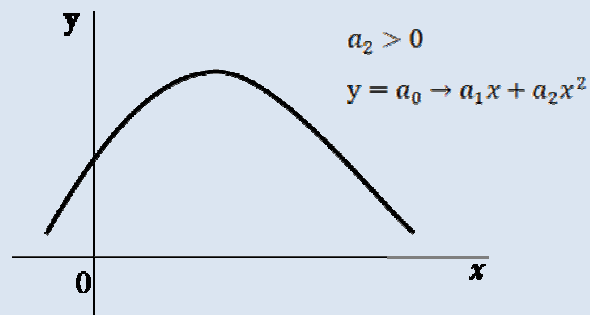


Fig 3.2A

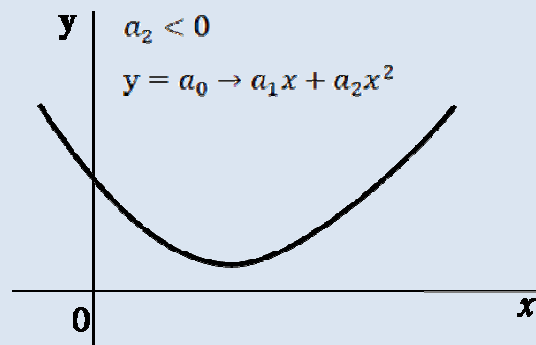


Fig 3.2B



d. When the highest power of the independent variable = 3 we have a cubic function → third-degree polynomial

i.e.

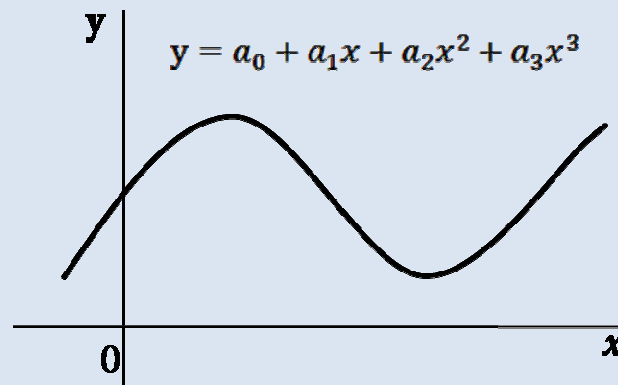


Fig.3.4

- **ITQ** Look at the graphs in Figure 3.5. Which of the two graphs indicates a constant function? What is wrong in the other graph?

Figure 3.5A

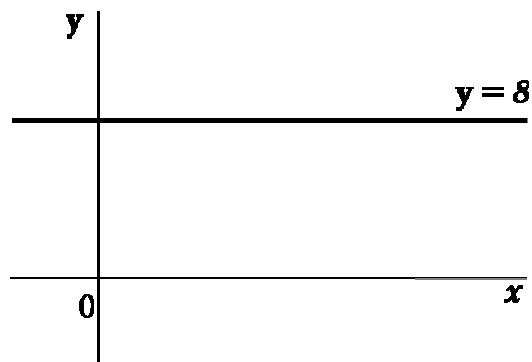
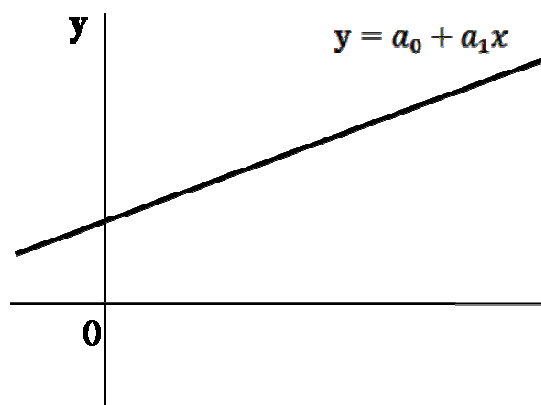


Figure 3.5B



Feedback:



Figure 3.5A indicates a constant function since the highest power of x , the independent variable is 0 and the curve is a horizontal.

The horizontal axis in Figure 3.5B is not labelled.

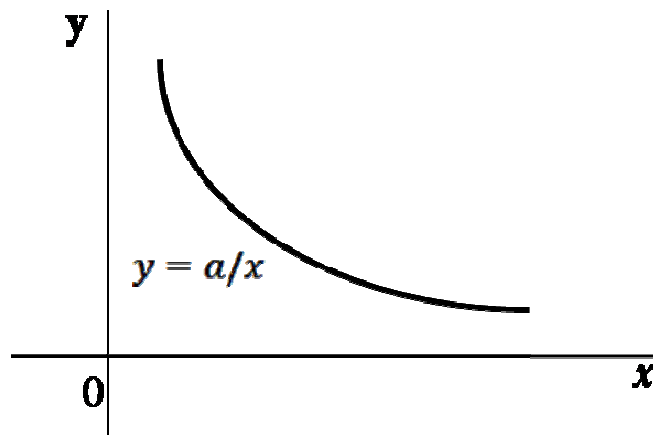
Rational Functions

A rational function is a function that is expressed as a ratio of 2 polynomials in the same variable, say x , i.e.

$$\frac{p(x)}{q(x)}$$

A rational function $\rightarrow y = \frac{a}{x}$ or $y = a/x$ plots as rectangular hyperbola as you can see in Fig 3.6

Figure 3.6
Rectangular-hyperbola



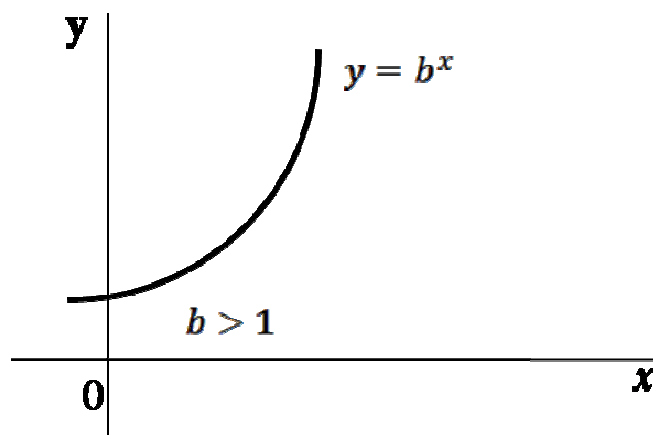
Transcendental Functions

This is a form of functions in which the independent variable appears in a nonalgebraic form. As such, another name for transcendental function is nonalgebraic function / In general, the term transcendental means nonalgebraic.

E.g. in exponential function:

independent variable appears in the exponent

Figure 3.7
Exponential





*Exponents show the power to which a particular variable is raised.
is raised to the power .*

Study session summary



Summary

In this Study session you learned how to construct functions. You were also exposed to the different types of functions and how they are graphed. The different types of functions include constant function which is graphed linearly; polynomial function which can be quadratic or cubical in graph, rational function which appears as rectangular-hyperbolic in graph; and lastly, transcendental function which is exponential.

Assessment



Assessment

SAQ 3.1 (tests Learning Outcome 3.1)

Construct one linear function and one cubic function.

SAQ 3.2 (tests Learning Outcome 3.2)

What do you understand by non algebraic functions and constant functions?

SAQ 3.3 (tests Learning Outcome 3.3)

Graph the following functions and see an instructor for evaluation.

1. $y = 8$
2. $y = \beta_0 + \beta_1 x$
3. $y = 2 + \alpha_1 x$
4. $y = \alpha_0 - \alpha_1 x + \alpha_2 x^2$
5. $y = \alpha_0 - \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$



Study Session4

Matrix Algebra

Introduction

In this study session, you will concept of matrix algebra, types of matrices, operation of matrices. Upon completion of this Study session you will be able to:



Learning Outcomes

- explain matrix algebra and the different types of matrices
- solve operations on matrices

Matrix

A matrix is a rectangular array of numbers, elements or variables arranged in rows and columns. The dimension or order of a matrix is determined / given by the number of rows and columns. So a matrix that has M rows and N columns has the dimension M X N.

* the row number must always precede the column number.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ a_{M1} & a_{M2} & a_{M3} & \dots & a_{MN} \end{pmatrix}$$

where a_{ij} is the element appearing in the i th row and j th column of A.

A has M rows and N columns \rightarrow dimension = M x N

$$B = \begin{pmatrix} 3 & 7 \\ 8 & 2 \\ 6 & 1 \end{pmatrix}$$

3 x 2

$$C = \begin{pmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \end{pmatrix}$$

2 x 3

When a matrix has only one column, it is referred to as a column vector:



$$A = \begin{pmatrix} 1 \\ 4 \\ 5 \\ 6 \end{pmatrix} \quad \text{dimension of } A = 4 \times 1$$

When a matrix contains only one row, it is called a row vector

$$B = \begin{pmatrix} 8 & 5 & 3 & 2 \end{pmatrix} \quad \text{dimension of } B = 1 \times 4$$

Types of Matrices

1. Square Matrix

A matrix with the same number of rows and columns.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 0 \end{pmatrix}$$

2. Diagonal Matrix

A square matrix that has at least one non zero element on the main diagonal and zeros elsewhere.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

3. Identity or Unit Matrix

A diagonal matrix whose diagonal elements are all 1. denoted by I

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Scalar Matrix

– A diagonal matrix whose diagonal elements are all equal.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$



5. Null Matrix

- A matrix whose elements are all zero and is denoted by 0

6. Null Vector

- A row or column vector whose elements are all zero, also denoted by 0

7. Equal Matrices

- 2 matrices A and B are equal if they are of the same order and their corresponding elements are equal.

$$A = \begin{pmatrix} 6 & 7 & 5 \\ 4 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 7 & 5 \\ 4 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix}$$

ITQ

Question

Which of the following matrices cannot be considered as a diagonal matrix?

- A square matrix
- A scalar matrix
- An identity matrix
- B and C

Feedback

The right option is a because though all diagonal matrices are square matrices, not all square matrices are diagonal matrices. For example, A 2×2 matrix may have no 0 on its off diagonals.

ITQ

Question

If matrix A is a 3×2 matrix, we mean _____.

Feedback

Matrix A has three rows and 2 columns, such that '×' means 'by'

Operation of Matrices

Addition of Matrices

The addition of 2 matrices A and B is possible if they are of the same order and addition is done by adding the corresponding elements of A and B.

e.g. if



$$A = \begin{pmatrix} 2 & 4 & 7 \\ 6 & 8 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 3 & 0 & 5 \end{pmatrix}$$

Then if $C = A + B$

$$C = \begin{pmatrix} 3 & 4 & 4 \\ 9 & 8 & 8 \end{pmatrix}$$

Subtraction of Matrices

Similar to the rule for addition, the subtraction of 2 matrices is only possible if they are of the same dimension.

$$A = \begin{pmatrix} 2 & 4 & 7 \\ 6 & 8 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 3 & 0 & 5 \end{pmatrix}$$

$$C = A - B$$

$$C = \begin{pmatrix} 1 & 4 & 10 \\ 3 & 8 & -2 \end{pmatrix}$$

Scalar Multiplication

To multiply a matrix A by a scalar (α)

We multiply each element of the matrix by α :

e.g. if $\alpha = 4$

$$\text{and } A = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 8 & 2 \end{pmatrix}$$

$$\alpha A = 4 \begin{pmatrix} 16 & 20 \\ 12 & 32 & 8 \end{pmatrix}$$

Multiplication of Matrices

For the multiplication of 2 matrices A and B to be possible, the number of columns in A must be equal to the number of rows in B and the resultant matrix AB will have the dimension where its no. of rows corresponds to no. of rows of A and its no. of columns corresponds to no. of columns of B.

e.g. if A dimension = M X N and B dimension = P X Q



multiplication is only possible if $N = P$

AB dimension $M \times Q$

If the product of the 2 matrices A and B is called C, then the element in the i th row and j th column of C is obtained by multiplying the elements of the i th row of A by the corresponding elements of the j th column of B and summing over all terms.

$$\text{e.g. given } A = \begin{pmatrix} 2 & 6 & 5 \\ 4 & 3 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 3 & 1 \\ 4 & 5 \\ 7 & 2 \end{pmatrix}$$

$$AB = C = \begin{pmatrix} (2 \times 3) + (6 \times 4) + (5 \times 7) & (2 \times 1) + (6 \times 5) + (5 \times 2) \\ (4 \times 3) + (3 \times 4) + (1 \times 7) & (4 \times 1) + (3 \times 5) + (1 \times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 65 & 42 \\ 31 & 21 \end{pmatrix}$$

$$\text{Given } A = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} -1 & 0 \\ 4 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} (3 \times (-1)) + (5 \times 4) & (3 \times 0) + (5 \times 7) \\ (4 \times (-1)) + (6 \times 4) & (4 \times 0) + (6 \times 7) \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 35 \\ 20 & 42 \end{pmatrix}$$

Properties of Matrix Multiplication

1. Matrix multiplication is not necessarily commutative i.e. in general $AB \neq BA$
2. Even if AB and BA exist the resulting matrices may not be of the same order

If A is $M \times N$ and B is $N \times M$

AB is $M \times M$

BA is $N \times N$

3. Even if A and B are both square matrices, so that AB and BA are both defined, the resulting matrices will not be necessarily equal.



$$A = \begin{pmatrix} 4 & 7 \\ 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 6 \\ 8 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 46 & 76 \\ 15 & 31 \end{pmatrix} \quad BA = \begin{pmatrix} 19 & 17 \\ 48 & 58 \end{pmatrix}$$

$AB \neq BA$

ITQ

Question

Matrix A (3×1) and matrix B (1×2) are commutative for multiplication since the number of columns in A equals the number of rows in B. What will be the order of matrix AB?

Feedback

(3×2), that is, the number of rows in A and the number of columns in B

Study session summary



Summary

In this Study session you were exposed to concept of matrix algebra, types of matrices, operation of matrices

Assessment

SAQ 4.1 (tests Learning Outcome 4.1)

Write out an example each of

- A matrix with 3 rows and two columns
- A square matrix of any dimension.
- A column vector
- A row vector
- An identity matrix
- A scalar matrix



SAQ 4.2 (tests Learning Outcome 4.2)

If

$$A = \begin{pmatrix} -2 & 5 & 4 \\ 3 & 0 & -6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -5 & 2 & 6 \\ 5 & 7 & 1 \end{pmatrix}$$

- a. What is $A + B$?
- b. Find $A - B$
- c. Solve for $3A$
- d. Why is it impossible to calculate AB with matrices A and B ?
- e. Find AC if

$$C = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$



Study Session 5

Linear-Equation System

Introduction

In the previous session, you were introduced to the concept of matrix algebra, types of matrices and operation of matrices. In this study session, you will be exposed to how to

Upon completion of this Study session you will be able to:



Learning Outcomes

- *represent linear equation systems in matrix form*
- *find the transpose and determinant of a matrix*
- *identify different economic applications of matrices*

Solving Linear Systems Using Matrices

Given the following linear equation system:

$$6x_1 + 3x_2 + x_3 = 22$$

$$x_1 + 4x_2 - 2x_3 = 12$$

$$4x_1 + x_2 + 5x_3 = 10$$

We can write these in matrix form as:

$$\begin{pmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 22 \\ 12 \\ 10 \end{pmatrix}$$

If we represent first matrix by A, second by x and third by d, we have:

$$Ax = d$$

$$\text{And } x = A^{-1} d$$

Therefore, the solution of the system of equation lies in finding the inverse of the matrix A

Transposes

The transpose of a matrix is the matrix obtained when its rows and columns are, interchanged i.e. first row is now the first column, second row is now the second column ...

Transpose of A is denoted by A^1 or A^T



$$\text{Given } A = \begin{pmatrix} 2 & 8 & 0 \\ 4 & 10 & 6 \\ 7 & 1 & 5 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 2 & 4 & 7 \\ 8 & 10 & 1 \\ 0 & 6 & 5 \end{pmatrix}$$

If we have a square matrix whose transpose gives exactly the same matrix, we have a Symmetric Matrix.

Properties of Transposes

1. $(A^1)^1 = A$
2. $(A + B)^1 = A^1 + B^1$
3. $(AB)^1 = B^1 A^1$
- 4.

ITQ

Question

If $A = \begin{bmatrix} 2 & 1 & 0 & 6 \end{bmatrix}$, the transpose of A will be

Feedback

$A^1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 6 \end{bmatrix}$ In other words, the transpose of a row vector is a column vector and vice versa.

Inverses

Given a matrix A, the inverse of A exists if and only if

1. A is a square matrix \rightarrow necessary condition
2. The rows are linearly independent \rightarrow sufficient condition

Linear independence means that no row (row vector) is a linear combination of other rows (row vectors).

$$\text{e.g. } A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{pmatrix} \quad \begin{pmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{pmatrix}$$



$$\begin{bmatrix} 6 & 8 & 10 \\ 2 & 2 & 0 \end{bmatrix} = 2 \begin{bmatrix} 4 & 5 \\ + & 0 \end{bmatrix}$$

$$V_3^1 = 2 V_1^1 + 0 V_2^1$$

Therefore, 3rd row can be expressed as linear combination of first 2 – rows are not linearly independent.

→ if rows in matrix are not linearly independent, no unique solution can be obtained →

*Given the square matrix A, if it's inverse exists, then A is called a nonsingular matrix. Conversely, if its inverse does not exist, A is a singular matrix.

note: $AA^{-1} = A^{-1}A = I$

If $A = p \times p$ then $A^{-1} = p \times p$

Determinants

Determinants are used to ascertain if a square matrix is nonsingular or not. The rule is that a matrix A is nonsingular if its determinant

$|A|$ is non-zero

2nd order determinant

$$A = \begin{pmatrix} 8 & 6 \\ 2 & 3 \end{pmatrix}$$

$$|A| = 8(3) - 2(6)$$

$$= 24 - 12$$

$$= 12$$

Expansion of higher order determinants can be done by the Laplace Expansion which is expansion by multiplying a sub-determinant by a row or column element.

Given

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

*Given linear equation system $Ax = d$

$|A| \neq 0 \rightarrow$ row/column independence

\rightarrow A is nonsingular

$\rightarrow A^{-1}$ exists

\rightarrow a unique soln $\bar{X} = A^{-1}d$ exists.

Expansion using the first row of A gives



$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

\uparrow Minor or element a_{11} \uparrow minor of element a_{12} \uparrow minor of element a_{13}
 $= |M_{11}|$ $= |M_{12}|$ $= |M_{13}|$
 $|C_{11}|$ $|C_{12}|$ $|C_{13}|$

Cofactor is minor with a given sign \rightarrow if sum of 2 subscripts is even, cofactor has same sign as minor. If sum of 2 subscripts is odd, cofactor has different sign as minor.

ITQ

Question

The minor of element a_{11} in matrix $X = \begin{pmatrix} 3 & 7 & 8 \\ 3 & 4 & 2 \\ -2 & 1 & 0 \end{pmatrix}$ is

Feedback

$$\begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix}$$

Given $A = \begin{pmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{pmatrix}$

First row

$$|A| = 5 \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 6 \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix}$$

$$= 0 + 0 - 27 = -27$$

First column

$$|A| = 5 \begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 2 \begin{vmatrix} 6 & 1 \\ -3 & 0 \end{vmatrix} + 7 \begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 0 - 6 - 21 = -27$$



$$B = \begin{pmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & -1 \end{pmatrix}$$

$$|B| = 7 \begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix} - (-3) \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} + (-3) \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= 7(-2) + 3(-2) - 3(-4)$$

$$= -14 - 6 + 12$$

$$= -8$$

* Solving these types of problems can be done in 2 ways:

1. - Matrix Inversion
2. - Cramer's Rule

Study session summary



Summary

In this study session you used matrices to find solution to a linear system of equations.



Assessment



Assessment

SAQ 5.1 (tests Learning Outcome 5.1)

Represent the linear equations below in matrix form

$$5x_1 + 3x_2 + x_3 = 20$$

$$-x_1 + 7x_2 - 2x_3 = 12$$

$$3x_1 - 5x_3 = 12$$

SAQ 5.2 (tests Learning Outcome 5.2)

Find the transpose and the determinant of each of the following matrices:

$$A = \begin{pmatrix} 5 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$$

SAQ 5.3 (tests Learning Outcome 5.3)

Prove that matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ is a singular matrix



Study Session6

System of Equations

Introduction

In Study Session 4, you were introduced to the concept of matrix algebra, types of matrices and operation of matrices; while you were exposed to how to use matrices to find solution to a linear system of equations in the just concluded study session. You will now learn how to find the solution of a system of equation using matrices, and how to apply matrices to economic models.

Upon completion of this Study session you will be able to:



Learning Outcomes

- solve problems relating to system of equation using matrices.
- apply matrices to economic models.

Finding the Solution of a System of Equations

We saw earlier that given a linear equation system such as:

$$6x_1 + 3x_2 + x_3 = 22$$

$$x_1 + 4x_2 - 2x_3 = 12$$

$$4x_1 - x_2 + 5x_3 = 10$$

can be rewritten in matrix form as:

$$\begin{pmatrix} 6 & 3 & 1 \\ 1 & 4 & -1 \\ 4 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 22 \\ 12 \\ 10 \end{pmatrix}$$

$\begin{matrix} A & x & = & d \end{matrix}$

And $x = A^{-1} d$ * up ↑

ITQ

Question

In the general matrix form of equation in subsection 6.1, $x = A^{-1} d$, what do the “x” and “d” stand for?

Feedback

X is the vector of unknowns while d is a vector of dependent variables



Matrix Inversion

To find the inverse of a matrix A , we need both the determinant and adjoint of A . The adjoint of A is simply the transpose of a cofactor matrix (C).

A cofactor denoted by $|C_{ij}|$ is a minor with a prescribed algebraic sign attached to it.

Rule \rightarrow if sum of subscripts in minor M_{ij} is even \rightarrow cofactor has the same sign as minor

If sum of subscripts of minor is odd \rightarrow cofactor has the opposite sign as minor

$$\text{i.e. } |C_{ij}| = (-1)^{i+j} |M_{ij}|$$

e.g.

$$\text{Given } A = \begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\text{Minor of element 8 is } |M_{12}| = \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = -6$$

$$\text{Cofactor of element 8 is } |C_{12}| = - |M_{12}| = 6$$

Because $i + j = 1 + 2 = 3$ is odd

$$\text{Also, cofactor of element 4 is } |C_{23}| = - |M_{23}| = -9 \begin{vmatrix} 8 & 7 \\ 3 & 2 \end{vmatrix} = 6$$

e.g.

$$\text{Given the matrix } A = \begin{pmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{pmatrix}$$



The cofactor matrix is

$$C = \begin{pmatrix} 4 & -2 & 1 & -2 & 1 & 4 \\ -1 & 5 & 4 & 5 & 4 & -1 \\ 3 & 1 & 6 & 1 & 6 & 3 \\ -1 & 5 & 4 & 5 & 4 & -1 \\ 3 & 1 & -6 & 1 & 6 & 3 \\ 4 & -2 & 1 & -2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & -13 & -17 \\ -16 & 26 & 18 \\ -10 & 13 & 21 \end{pmatrix}$$

The inverse of a matrix A is given as:

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{|A|} C^T$$

Where $|A|$ = determinant of A

Adj A = adjoint of A

C^T = transpose of the cofactor matrix C

Steps in Inverting a Matrix A

1. find $|A|$
2. find cofactors of all elements of A and arrange them as a cofactor matrix $C = [C_{ij}]$
3. take transpose of C to get adj A

4. divide adj A by determinant $|A|$

e.g. find inverse of $A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$



$|A| = -2 \neq 0$ inverse of A exists

$$C = \begin{pmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}$$

Transpose C

$$\text{Adj } A = \begin{pmatrix} 0 & -2 \\ -1 & 3 \end{pmatrix}$$

Then

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{-1}{2} \begin{pmatrix} 0 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Find inverse of $B = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{pmatrix}$ $|B| = 99$

Cofactor matrix is

$$21 \begin{pmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & - \begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ - \begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & - \begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{pmatrix} = -7 \begin{pmatrix} 31 & 3 \\ -8 & 12 \end{pmatrix}$$

$$\text{Adj } B = \begin{pmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{pmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{99} \begin{pmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{pmatrix}$$



Given a system of equations, the solution lies in multiplying the inverse B^{-1} by the column vector d

Cramer's Rule

Cramer's rule gives the rule for finding the solution of a system of equations as

$$\begin{array}{ccccccc} a_{11} & a_{12} & \dots & d_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & d_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & d_n & \dots & a_{nn} \end{array}$$

$\bar{X}_j = \frac{|A_j|}{|A|} = \frac{1}{|A|}$

(jth column replaced by d)

$$\bar{X}_1 = \frac{|A_1|}{|A|}$$

$$\bar{X}_2 = \frac{|A_2|}{|A|}$$

Cramer's rule involves first finding the determinant. To get the solution value of the jth variable x_j we replace the jth column of the determinant by the constant terms $d \dots d_n$. This gives a new determinant denoted by $|A_j|$. The new determinant $|A_j|$ is now divided by original determinant $|A|$ to give the value of x_j

Note

→ matrix inversion method gives the solution values of all endogenous variables at once (\bar{X} is a vector)

→ Cramer's rule can give us the solution value of only a single endogenous variable at a time (\bar{X}_j is a scalar).

e.g.

(1) find the solution of the equation system

$$\begin{array}{l} 5x_1 + 3x_2 = 30 \\ 6x_1 - 2x_2 = 8 \end{array} \quad \begin{pmatrix} 5 & 3 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28$$



$$|A_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84$$

$$|A_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

$$\bar{X}_1 = \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \quad \bar{X}_2 = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5$$

(2) find the solution of the equation system

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned} \quad \begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{matrix} 0 \\ 8 \\ 7 \end{matrix}$$

$$|A| = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{vmatrix} = -61$$

$$|A_1| = \begin{vmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{vmatrix} = -61$$

$$|A_2| = \begin{vmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{vmatrix} = -183$$

$$|A_3| = \begin{vmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{vmatrix} = -244$$

$$\bar{X}_1 = \frac{|A_1|}{|A|} = \frac{-61}{-61} = 1 \quad \bar{X}_2 = \frac{|A_2|}{|A|} = \frac{-183}{-61} = 3$$

$$\bar{X}_3 = \frac{|A_3|}{|A|} = \frac{-244}{-61} = 4$$

**ITQ****Question**

In relation to the solution to variables, what is the difference between Cramm's rule and Matrix inversion method?

Feedback

Cramm's rule presents the solution value of only one unknown variable at a time while matrix inversion solves for the values of all the unknown variables at once.

Economic Applications of Matrices

1. Matrix algebra can be used for solving different models in economics. Such models include general equilibrium models, Bordered Hessians and Jacobian models
2. Matrices can be used for finding the equilibrium price and quantities market models e.g. demand and supply functions.
3. Matrix algebra is also useful for solving National Income models.

Knowledge of the use of matrices in solving economic models will be acquired in higher levels of economic courses.



Assessment



Assessmen

SAQ 6.1 (tests Learning Outcome 6.1)

a. Find the inverse of the following matrix:

$$A = \begin{pmatrix} 7 & 6 \\ 0 & 3 \end{pmatrix}$$

b. Use Cramer's rule to solve the following systems of equation:

$$3x_1 - 2x_2 = 11$$

$$2x_1 + x_2 = 12$$

SAQ 6.2 (tests Learning Outcome 6.2)

Highlight 3 economic models where matrices are applicable?



Study Session 7

Differential Calculus and its Application in Economics

Introduction



Learning Outcomes

In this study session, you will examine the concept of differential calculus. However, before that, I shall take you through some conceptual clarification of the concept of limit and the derivative.

Upon completion of this Study session you will be able to:

- *explain* the derivative of a function.
- *apply* the different rules of differentiation.
- *apply* the concept of differentiation to Economics.

Rate of Change and the Derivative of a Variable

In differential calculus, we can consider the rate of change of a particular variable y in response to a change in another variable x , under the condition that the two variables (y and x) are related to each other by the function.

$$Y = f(x)$$

The variable y can be referred to as the endogenous variable and x the exogenous variable. This is just an example of the simple case where there is only a single parameter or exogenous variable in the model. The different extensions to the case of more than one exogenous variable in the model can then be made once we have mastered this simplified case.

When the variable x changes from the value x_0 to a new value x_1 , the change is measured by the difference $x_1 - x_0$. We can then use the symbol Δ (the Greek capital delta, for difference) to define the change. We can therefore write

$$\Delta x = x_1 - x_0$$

In addition, we can denote the value of the function $f(x)$ at various values of x . This can be done by using the notation $f(x_i)$ to represent the value of $f(x)$ when $x = x_i$. Therefore, for the function $f(x) = 2 + x^2$, we can determine it as $f(0) = 2 + 0^2 = 2$; and similarly, $f(2) = 2 + 2^2 = 6$.

Furthermore, it should be noted that when x changes from an initial value x_0 to a new value $(x_0 + \Delta x)$, the value of the function $y = f(x)$ in equation 1 will also change from $f(x_0)$ to $f(x_0 + \Delta x)$. The change in y per unit of change in x can be represented by the difference quotient.



$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This formula measures the average change of y .

In most cases, we are interested in the rate of change of y when Δx is very small. In such a situation, we can obtain an approximation of $\Delta y/\Delta x$ by dropping all the terms in the difference quotient involving the expression Δx . In this case, the smaller the value of Δx , the closer is the approximation to the true value of $\Delta y/\Delta x$. This can be expressed in symbolic term as:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The symbol $\lim_{\Delta x \rightarrow 0}$ can be interpreted as the limit of the function in the equation as Δx approaches 0. If as Δx tends toward zero ($\Delta x \rightarrow 0$), the limit of the difference quotient $\Delta y/\Delta x$ exists. That limit is identified as the derivative of the function $y = f(x)$.

It should however, be noted that a derivative is a function, that is a derived function. The original function $y = f(x)$ is a primitive function, and the derivative is another function derived from it. In addition, since the derivative is a limit of the difference quotient, the derivative must also be a measure of some rate of change. Derivatives can also be denoted by using the symbol $f'(x)$ or dy/dx . Combining these two notations, we may define the derivative of a given function $y = f(x)$ as:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Rules of Differentiation

It is necessary to first discuss some of the rules that apply to some types of function of a single independent variable: $y = k$ (constant function), $y = x^m$ and $y = bx^m$ (power functions)

Differentiation of a Constant Function Rule

The derivative of a constant function $y = f(x) = k$ is zero for all values of x . We can express this in symbolic terms as:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dk}{dx} = 0 \quad f'(x) = 0$$

Example: Given $y = 63$, we have $dy/dx = 0$ or $f'(y) = 0$

Example: Given $x = 40$, we have $f'(x) = 0$ or $dy/dx = 0$

Therefore, the differentiation of a constant function is zero for all values of x .



Differentiation of a Power-Function Rule

The derivative of a power function $y = f(x) = X^m$ is mx^{m-1}

In symbolic terms we can express this as:

$$\frac{d}{dx} x^m = mx^{m-1} \quad \text{or} \quad f'(x) = mx^{m-1}$$

dx

Example: The derivative of $y = x^5$ is $dy/dx = \frac{d}{dx} x^5 = 5x^{5-1} = 5x^4$

dx

Example: The derivative of $y = x^9$ is $dy/dx = \frac{d}{dx} x^9 = 9x^{9-1} = 9x^8$

dx

Example: Find the derivative of $y = x^0$. We can apply the power function rule to find

$$\frac{d}{dx} x^0 = 0(x^{-1}) = 0$$

dx

Example: Find the derivative of $y = \frac{1}{x^2}$. The process of differentiating this

function involves the reciprocal of a power. However, rewriting the function as $y = x^{-2}$, we can apply the power function rule to arrive at the derivative:

$$\frac{d}{dx} x^{-2} = -2 x^{-2-1} = -\frac{2}{x^3}$$

dx

Example: Find the derivative of $y = \sqrt{x}$. A square root is involved in this function. However, we can rewrite the function as $\sqrt{x} = x^{1/2}$, the derivative can be found as follows:

$$\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

dx

Derivatives are functions of the independent variables x themselves. For example, the derivative in the example $dy/dx = 5x^4$ such that a different value of x will result in a different value of the derivative, such as

$$f'(1) = 5(1^4) = 5 \quad f'(2) = 5(2^4) = 80$$

Alternatively, the specific values can be expressed as:

$$\left. \frac{dy}{dx} \right|_{x=1} = 5 \quad \left. \frac{dy}{dx} \right|_{x=2} = 80$$

It is necessary to first differentiate the function $f(x)$ in order to get the derivative function $f'(x)$ and thereafter let x assume specific values in $f'(x)$.

ITQ

Question

If I differentiate a function and the derivative of the function equals zero,



what type of function did I differentiate?

Feedback

It is a constant function. The derivatives of constant functions are always zero.

Differentiation of a Generalized Power Function Rule

When a multiplicative constant k appears in the power functions, so that $f(x) = kx^m$, its derivative can be written as:

$$\frac{d}{dx} kx^m = kmx^{m-1} \quad \text{or} \quad f'(x) = kmx^{m-1}$$

This formula shows that, in differentiating kx^m , we can retain the multiplicative constant k intact and then differentiate the term x^m according to the power function rule.

Example: Given $y = 4x$, we have $dy/dx = 4x^{1-1} = 4x^0 = 4$

Example: Given $f(x) = 6x^3$, the derivative is $f'(x) = 6(3)x^{3-1} = 18x^2$

Example: The derivative of $f(x) = 4x^{-2}$ is $f'(x) = 4(-2)x^{-2-1} = -8x^{-3}$

ITQ

Question

What is the difference between a power function rule and a generalised power function rule?

Feedback

The later takes into account the presence of a multiplicative constant in the function being differentiated.

Differentiation of a Sum – Difference Function Rule

The derivative of a sum or difference of two functions is the sum or difference of the derivatives of the two functions.

If the function is a sum:

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) = f'(x) + g'(x)$$

If the function is a difference:

$$\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x) = f'(x) - g'(x)$$



Example: Given $y = 2x^3 + 13x^2 - x^3$. The derivative of this function according to the sum difference rule, is:

$$\frac{dy}{dx} = \frac{d}{dx} (3x^2 + 13x^2 - x^3) = 6x + 26x - 3x = 30x$$

Example: Given the function $\frac{d}{dx} (ax^2 + bx + c)$

The derivative is $\frac{d}{dx} (ax^2 + bx + c) = 2ax + b$

Example: Given the function $7x^3 + 2x^2 - 3x + 35$

$$\frac{d}{dx} = 21x^2 + 4x - 3$$

In the last two examples, the constant k and 35 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to the multiplicative constant, which is retained during differentiation, the additive constant drops out.

Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function:

$$\begin{aligned} \frac{d}{dx} [f(x) g(x)] &= f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \\ &= f(x) g'(x) + g(x) f'(x) \end{aligned}$$

Example: Find the derivative of $y = (2x + 3) (3x^2)$

Let $f(x) = 2x + 3$ and $g(x) = 3x^2$. Therefore, $f'(x) = 2$ and $g'(x) = 6x$

$$\frac{d}{dx} [(2x + 3) (3x^2)] = (2x + 3) (6x) + (3x^2)(2) = 18x^2 + 18x$$

Quotient Rule

The derivative of the quotient of two functions, $f(x)/g(x)$, is:

$$\frac{df(x)}{dx g(x)} = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}$$



Example: $\frac{d}{dx} \frac{2x-3}{x+1}$

$$\frac{d}{dx} \frac{2x-3}{x+1}$$

Let $f(x) = 2x - 3$; $g(x) = x + 1$; $f'(x) = 2$; $g'(x) = 1$ $g^2(x) = (x + 1)^2$

$$\frac{2(x+1) - (2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

Example given: $\frac{d}{dx} \frac{5x}{x^2+1}$

$$\frac{d}{dx} \frac{5x}{x^2+1}$$

$F(x) = 5x$; $g(x) = x^2 + 1$; $f'(x) = 5$; $g'(x) = 2x$; $g^2(x) = (x^2 + 1)^2$

$$\frac{d}{dx} \frac{5x}{x^2+1} = \frac{5(x^2+1) - 5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

Chain Rule

If we have a function $z = f(y)$, where y is in turn a function of another variable x , say, $y = g(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y , times the derivative of y with respect to x .

This can be expressed as:

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(y) g'(x)$$

This rule is known as the chain rule (function of a function)

Example: If $z = 3y^2$, where $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$

$$\begin{aligned} \frac{dz}{dy} &= 6y \quad \frac{dy}{dx} = 2 \\ &= 6y(2) = 12y = 12(2x + 5) \end{aligned}$$



ITQ

Question

If $y = 4(5x + 7)^2$, which rule can be easily applied besides the product rule?

Feedback

The chain rule. By denoting $(5x + 7)$ by any letter, say z , one can first derive dy/dz before deriving dy/dx and multiplying both derivatives.

Some Economic Applications of Differentials

As an illustration of the application of differentials in economics, let us consider the concept of revenue and cost.

For example, if we are given an average – revenue (AR) function in specific for

$$AR = 10 - Q$$

The marginal revenue (MR) function can be found by first multiplying AR by Q to get the total-revenue (TR) function:

$$TR = AR \cdot Q = (10 - Q)Q = 10Q - Q^2$$

And then differentiating TR to yield the MR

$$MR \equiv \frac{dTR}{dQ} = 10 - 2Q$$

The average revenue can also derived as:

$$AR \equiv \frac{TR}{Q} \equiv \frac{P \cdot Q}{Q} \equiv P$$

With respect to Cost, the marginal cost (MC) can also be derived by differentiating the total cost (TC).

For example, given the total cost function:

$$TC = Q^3 - 10Q^2 + 40Q$$

The marginal cost can be derived as:

$$MC = \frac{dTC}{dQ} = 3Q^2 - 20Q + 40$$

The total cost can also be derived as:

$$AC = \frac{TC}{Q} = Q^2 - 10Q + 40$$



Study session summary



Summary

In this study session you learned how to:

- *explain* the derivative of a function.
- *apply* the different rules of differentiation.
- *apply* the concept of differentiation to Economics.

Assessment



Assessment

SAQ 7.1 (tests Learning Outcome 7.1)

What do you understand by the derivative of a function?

SAQ 7.2 (tests Learning Outcome 7.2)

Find the derivatives of the following functions:

- $y = 7x^2 + 3$
- $y = (3x + 2)(4x^2)$
- $y = (4x^2 - 7x)/3x^2$

SAQ 7.3 (tests Learning Outcome 7.3)

A firm has a revenue function of the type $TR = 900q - 50q^2$. If its total cost $TC = 420 + 210q - 60q^2 + 75q^3$, find the;

- Output for which profit is maximised
- Profit at that output level



Study Session 8

Integral Calculus

Introduction



Outcomes

This study session introduces the concept of integration. The basic relationship between integral calculus or integration and differentiation shall be explored. This will help in understanding the rules of integration, the different types of integrals and its application in Economics.

Upon completion of this Study session you will be able to:

- *explain* the integral of a function.
- *apply* the basic rules of integration.
- *apply* the concept of integration in Economics.

The Concept of Integral Calculus

In the previous lectures, we discussed the concept of differentiation, whereby our preoccupation was to find the values of the choice variables that maximize (or minimize) a specific objective function. In this present study session we are interested in tracing back the time path of some variables, given a known pattern of change (or a given instantaneous rate of change). Take for example, suppose that the population size H is known to change over time at the rate

$$\frac{dH}{dt} = t^{-1/2} \quad \text{-----(1)}$$

We then try to find what time path(s) of population $H = H(t)$ can yield the rate of change in (1). In other words, we try to trace the percentage of the function in (1). Eqn. (1) was found by differentiation, we are now confronted with the problem of uncovering the primitive function that was differentiated to produce eqn (1). Hence, we now need an exact opposite of the method of differentiation, or of differential calculus. This method is called integration, or integral calculus.

Given that the primitive function that produces eqn (1) is $H(t) = 2 t^{1/2}$, thus apparently qualifying as a solution to our problem. The concern is that there also exist similar functions, such as $H(t) = 2t^{1/2} + 15$ or $H(t) = 2t^{1/2} + 99$ or, more generally,

$H(t) = 2t^{1/2} + c \quad \text{.....(2)}$ (c = an arbitrary constant) which all yield the same derivative (i.e. eqn 1). The question therefore, is how do we determine for some, the exact primitive function that yielded eqn 1? The solution is not far fetched, an additional information must be provided



into the model, usually in the form of what is called the initial condition or boundary condition. Given that the initial condition is provided, the value of the constant c can be made determinate. Setting $t = 0$ in eqn (2) i.e. $H(0) = 0$, we get:

$$H(0) = 2(0)^{1/2} + c = c$$

But if $H(0) = 100$, then $c = 100$ and eqn (2) becomes:

$$H(9t) = 2t^{1/2} + 100.$$

Note that $H(0)$ refers to the value of H at $t = 0$, i.e. at the initial period.

Thus, for any given initial population $H(0)$, the time path will be

$$H(t) = 2t^{1/2} + H(0).$$

Hence, the population size H at any point of time will, in the present example consist of the sum of the initial population $H(0)$ and another term involving the time variable t .

ITQ

Question

What is the likely difficulty of integrating an initially differentiated function without setting an initial or boundary condition?

Feedback

There can be several similar functions that can serve as the primitive function of a differential. For example, the integral of $14x$ is $7x^2$ (that is, when $7x^2$ is differentiated, we get $14x$). Without a boundary condition, $7x^2$ plus any constant will yield $14x$. So merely integrating $14x$ cannot yield a definite integral or primitive function without a boundary condition.

Indefinite Integrals

The Nature of Integrals

Integral is known to be the reverse of differentiation. If the differentiation of a given primitive function $F(x)$ yields the derivative $f(x)$, we can “integrate” $f(x)$ to find $F(x)$, given that the relevant information are provided to determine the arbitrary constant which will arise in the process of integration.

$F(x)$ is referred to as an integral of the function $f(x)$.

The standard notation used to denote integral is \int

Thus, the integration of $f(x)$ with respect to x is $\int f(x)dx$

Where \int - is the integral sign

$f(x)$ – is the integrand (function to be integrated)



dx – reminds us that the operation is to be performed with respect to x

Note that $f(x) dx$ can also be taken as a single entity and interpreted as the differential of the primitive function $F(x)$ [i.e. $dF(x) = f(x)dx$].

The integral sign in front gives the instruction for the reversal of the differentiation process that gives rise to the differential.

$$\text{Thus, } dF(x) = f(x) \implies \int f(x)dx = F(x) + c \quad \dots\dots(3)$$

dx

The integral $\int f(x)dx$, is more specifically known as the indefinite integral of $f(x)$. Why? because it has no definite numerical value. Its value will in general vary with the value of x (even if c is definitised).

Basic Rules of Integration

Our knowledge of the rules of derivation in the previous study session will go a long way in understanding the basic rules of integration from the following derivative formula for a power function:

$$\frac{d}{dx} x^{n+1} = x^n \quad (\text{where } n \neq -1)$$

$dx \quad n+1$

The expression $\frac{x^{n+1}}{n+1}$ is the primitive function for the derivative function x^n ; thus by

$n+1$

Substituting these for $F(x)$ and $f(x)$ in eqn (3), we may have the result as a rule of integration.

Rule 1: Power Rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \quad (n \neq -1)$$

$n+1$

Example 1: find $\int x^2 dx$, here we have $n = 2$, and

therefore

$$\int x^2 dx = \frac{1}{3} x^3 + c$$

Example 2: Find $\int \frac{1}{x^3} dx$, ($x \neq 0$) since

x^3

$1/x^3 = x^{-3}$, we have $n = -3$. Thus, the integral is

$$\begin{aligned} \int 1/x^3 dx &= \frac{1}{-2} x^{-2} + c = -\frac{1}{2} \cdot \frac{1}{x^2} + c \\ &= -\frac{1}{2x^2} + c \end{aligned}$$



$$2x^2$$

Rule 2: Exponential rule

$$\int e^x dx = e^x + c$$

Rule 3: Logarithmic rule

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

x

This is a special form of power function where $n = -1$

Variants of rule 2 and 3 exist and they have the following rules:

Rule 2a

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + C.$$

Rule 3a

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c \quad [f(x) > 0]$$

F(x)

$$\text{or} \quad \ln |f(x)| + c \quad [f(x) \neq 0]$$

ITQ

Question

If I have a differential $dy/dx = x$, what will be the integral using the power rule of integration?

Feedback

$$\int x dx = \frac{1}{2}(x^2) + c$$

Rule 4: Integral of a sum

The integral of the sum of a finite number of functions is the sum of the integrals of those functions. In other words, for a two-function case, this implies:

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Following from the illustration given in eqn (3);

$$\frac{d}{dx} [F(x) + G(x)] = \frac{d}{dx} F(x) + \frac{d}{dx} G(x) = f(x) + g(x)$$

$$\underbrace{\int f(x) dx}_A \quad \underbrace{\int g(x) dx}_B \quad \underbrace{\int g(x) dx}_C$$

Given that $A = C$, which is true, then we can apply the same process used in eqn (3) thus,

$$\int [f(x) + g(x)] dx = F(x) + G(x) + c.$$



But the fact that we can split A, the function, into two, (B) implies that we can have two independent constant as in:

$$\int f(x) dx = F(x) + c_1 \quad \text{and} \quad \int g(x) dx = G(x) + c_2$$

Thus by addition, we have:

$$\int f(x) dx + \int g(x) dx = F(x) + G(x) + c_1 + c_2$$

Since the constants c_1 and c_2 are arbitrary in value, we can let $c = c_1 + c_2$.

This we have

$$\int f(x) dx + \int g(x) dx = F(x) + G(x) + c$$

Example 8.1

Find $\int (2x^3 + 3x + 1) dx$. By Rule 4, this integral can be expressed as a sum of three integrals:

$$= \int 2x^3 dx + \int 3x dx + \int 1 dx$$

$$= \left[\frac{2x^{3+1}}{3+1} + c_1 \right] + \left[\frac{3x^{1+1}}{1+1} + c_2 \right] + \left[\frac{1x^{0+1}}{0+1} + c_3 \right]$$

Note that $x^0 = 1$

$$= \frac{2x^4}{4} + \frac{3x^2}{2} + x + c_1 + c_2 + c_3$$

$$= \frac{x^4}{2} + \frac{3x^2}{2} + x + c \quad \text{where } c = c_1 + c_2 + c_3$$

Example 8.2

$$\text{Find } \int \left(3e^{3x} + \frac{15x}{5x^3 + 7} \right) dx$$

If we follow rule 4, we can integrate the two additive term in the integrand separately, and then sum the results. Since the $3e^{3x}$ term is in format $f'(x) e^{f(x)}$ in rule 2a, with $f(x) = 3x$, the integral is $e^{3x} + c_1$. Similarly, the other term, $15x/(5x^3 + 7)$, takes the form of $f'(x)/f(x)$, with $f(x) = 5x^3 + 7 > 0$. Thus, by rule 3a, the integral is $\ln(5x^3 + 7) + c_2$.

Hence we can write:

$$\int \left(3e^{3x} + \frac{15x}{5x^3 + 7} \right) dx = e^{3x} + \ln(5x^3 + 7) + c$$

Where $c = c_1 + c_2$.



ITQ

Question

The rule for the integral of a sum is applicable to the integral of a difference between two functions. Based on this, $\int [f(x) - g(x)] dx$ will be _____.

Feedback

$$\int f(x) dx - \int g(x) dx$$

Rule 5: Integral of a Multiple

$$\int k f(x) dx = k \int f(x) dx \text{ where } k \text{ is a constant.}$$

This means that, the integral of k times an integrand (where k is a constant) is k times the integral of integrand.

Example 8.3

Find $\int -f(x) dx$ where $k = -1$, and thus

$$\int -f(x) dx = - \int f(x) dx$$

Example 8.4

Find $\int 2x^3 dx$

$$\begin{aligned} \int 2x^3 dx &= 2 \int x^3 dx = \frac{2x^{3+1}}{3+1} + c \\ &= \frac{2x^4}{4} + c \\ &= \frac{1}{2} x^4 + c \end{aligned}$$

Rule 6: Substitution rule

The integral of $f(u)$ (du/dx) w.r.t. the variable x is the integral of $f(u)$ w.r.t. the variable u :

$$\int f(u) \frac{du}{dx} dx = \int f(u) du = F(u) + c$$

Proof

Given a function $F(u)$, where $U = u(x)$, the chain rule will be applied here and it states that:

$$\frac{d}{dx} F(u) = \frac{d}{du} F(u) \frac{du}{dx} = F'(u) \frac{du}{dx} = f(u) \frac{du}{dx}$$

Since the derivative of $F(u)$ is $f(u) (du/dx)$, then following from eqn (3), the integral of the derivative is $\int f(u) \frac{du}{dx} dx = F(u) + c$

dx

**Example 8.5**Find $\int 2(x^2 + 1) dx$

By simple rule, we can multiply the integrand

$$\begin{aligned}\int 2x(x^2 + 1) dx &= \int (2x^3 + 2x) dx = \frac{2x^{3+1}}{3+1} + \frac{2x^{1+1}}{1+1} + c \\ &= \frac{x^4}{2} + x^2 + c\end{aligned}$$

We can now apply the substitution rule that

 $U = x^2 + 1$; then $du/dx = 2x$ or $dx = du/2x$ If you substitute for dx , and u in the problem, we will have

$$\frac{du}{dx} = \int 2x \cdot u \cdot \frac{du}{2x} = \int u du$$

$$dx \quad 2x$$

$$\int u du = \frac{u^2}{2} + c_1 \quad [\text{substitute for } u]$$

$$\begin{aligned}&= \frac{1}{2} (x^2 + 1)^2 + c_1 = \frac{1}{2} (x^4 + 2x^2 + 1) + c_1 \\ &= \frac{x^4}{2} + x^2 + c\end{aligned}$$

$$= \frac{x^4}{2} + x^2 + c$$

[where $c = \frac{1}{2} + c_1$]**Example 8.5**Find: $\int 8e^{2x+3} dx$

Solution

Let $u = 2x + 3$; $du/dx = 2$, or $dx = du/2$

Hence

$$\int 8e^{2x+3} dx = \int 8e^u \frac{du}{2} = 4 \int e^u du = 4e^u + c = 4e^{2x+3} + c$$

2

[Note from Rule 2 that $\int e^u = e^u$]**Rule 7: Integration by parts**The rule states that the integral of v w.r.t. $-u$ is equal to uv less the integral of u w.r.t. $-v$.

$$\int v du = uv - \int u dv$$

Proof



If you remember from the product rule of differential where $d(uv) = vdu + u dv$

If we then integrate each differential, we get

$$\int d(uv) = \int vdu + \int u dv$$

$$Uv = \int vdu + \int u dv$$

By subtracting $\int u dv$ from both sides, we have

$$Uv - \int u dv = \int vdu.$$

Example 8.6

Find: $\int x e^{2x}$, using integration by parts.

Let $u = x$, $du = 1$; $dv = e^{2x}$, $\int dv = v \Rightarrow \int e^{2x} = \frac{1}{2} e^{2x}$

$$\begin{aligned} \text{Thus, } \int x e^{2x} dx &= x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \cdot 1 dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx + c = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c \end{aligned}$$

ITQ

Question

Given the function $y = 8x(x^2 + 3)^3$, which integral rule is applicable in integrating the function, integration by part or integration by substitution?

Feedback

Integration by substitution. This is because the integrand can be expressed as a constant multiple of another function u and its derivative, du/dx . That is, $x^2 + 3$ can serve as u and its derivative $du/dx = 2x$ can be used to divide $8x$ to get a constant multiple 4. As such a new integrand can be formed $\int 8x \cdot u \cdot du/2x = \int 4 \cdot u \cdot du$.

Definite Integrals

All the integrals cited in the preceding section are of the indefinite variety; each is a function of a variable and, hence, possesses no definite numerical value. Now for a given indefinite integral of a continuous function $f(x)$,



$$\int f(x) \, dx = F(x) + c.$$

If we choose two values of x in the domain, say a and b ($a < b$), substitute them successively into the right side of the equation, and form the difference.

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

We get a specific numerical value, free of variable x as well as the arbitrary constant c . This value is called the definite integral of $f(x)$ from a to b . a is referred to as the lower limit of integration and b , the upper limit of integration. The evaluation of the definite integral is then symbolized in the following steps.

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

Example 8.7

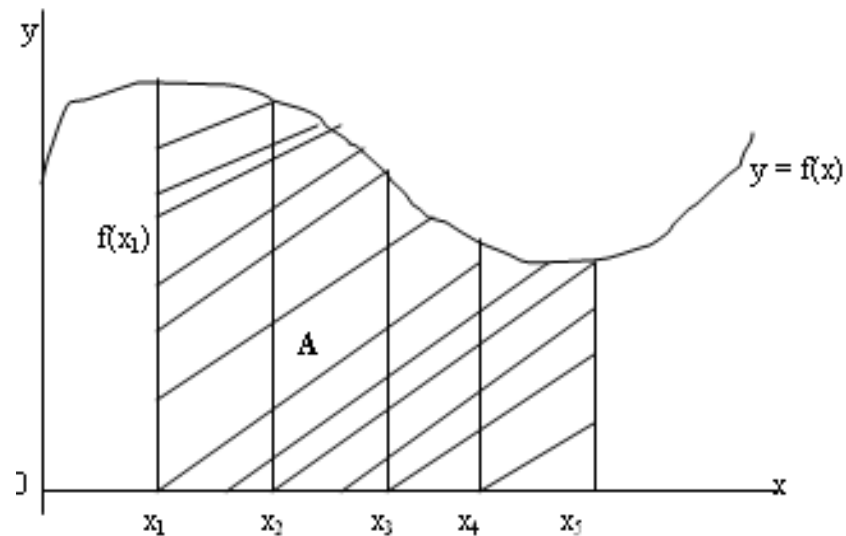
Evaluate $\int_a^b k e^x \, dx$

$$\int_b^a k e^x \, dx = k e^x \Big|_b^a = k(e^b - e^a)$$

The definite integral has a definite value. That value may be interpreted geometrically to be a particular area under a given curve.



The graph of a continuous function $y = f(x)$ is drawn in fig. 8.1



If we seek to measure the area A enclosed by the curve and the x axis between the two points a and b in the domain, we may proceed in the following manner.

1. We divide the interval $[a, b]$ into n sub intervals, we have four of such in fig. 4.1, i.e. $n = 4$ – the first being (x_1, x_2) and the last, (x_4, x_5) . Since each of these represents a change in x , we may refer to them as $\Delta x_1 \dots \Delta x_4$, respectively.
2. On the sub intervals, construct four rectangular blocks such that the height of each block is equal to the highest value of the function attained in that block (which occurs at the left-side boundary of each rectangle here). The first block has the height $f(x_1)$, and the width Δx_1 , and in general, the i th block has the height $f(x_i)$ and width Δx_i . The total area x (height x width x n) of this set of blocks is the sum.

$$A^* = \sum_{i=1}^n f(x_i) \Delta x_i \quad (n = 4 \text{ in fig. 8.1})$$

A^* is the approximate value of the true area A . It is approximate because it overestimates the actual value of A . But in the limit (i.e. as $n \rightarrow \infty$), A^* will approach the true value A .

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i = \lim_{n \rightarrow \infty} A^* = \text{area } A$$

How is eqn related to integration? You will notice that the summation expression in eqn; $\sum_{i=1}^n f(x_i) \Delta x_i$, bears a certain resemblance to the definite integral expression



$\int_a^b f(x) dx$. Indeed, the latter is based on the former.

When the Δx_i is infinitesimal (very small), we may replace it with the symbol dx .

The summation sign $\sum_{i=1}^n$ represents the sum of a finite number of terms. When we let $n \rightarrow \infty$, and take the limit of that sum, the symbol \int_a^b is used.

Thus, $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i = \text{Area } A$.

The definite integral (referred to as a Riemann integral) now has an area connotation as well as sum connotation because \int_a^b is the continuous counterpart of the discrete concept of $\sum_{i=1}^n$.

The overestimated area A^* , is approximated to the true area A , as $n \rightarrow \infty$ in the interval (a, b) , the resulting limit of the sum of block areas is called the upper integral. We could also approximate area A from below by forming rectangular blocks inscribed by the area rather than protruding beyond it like the earlier case. The total area A^{**} that ensue from this new set of blocks as $n \rightarrow \infty$ will underestimate the true area A . This last limit is called the lower integral. Hence the Riemann integral $\int_a^b f(x) dx$ is defined, if and only if, the upper and lower integral are equal in value and the function $f(x)$ is said to be Riemann integrable. According to the fundamental theorem of Calculus a function is integrable in $[a, b]$ if it is continuous in that interval.

Properties of Definite Integrals

Property 1

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

PII

$$\int_a^a f(x) dx = F(a) - F(a) = 0$$

PIII: additive property.

$$\int_a^d f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^d f(x) dx \quad (a < b < c < d)$$

PIV

$$\int_a^b -f(x) dx = - \int_a^b f(x) dx$$



PV

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx$$

PVI

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

PVII

$$\int_{x=a}^{x=b} v du = uv \Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} u dv$$

ITQ

Question

Evaluate the definite integral $\int_1^4 8x dx$.

Feedback

$$\int_1^4 8x dx = 4x^2 \Big|_1^4 = 4(4)^2 - 4(1)^2 = 64 - 4 = 60$$

Improper Integrals

Two varieties shall be discussed:

- Infinite Limits of Integration and
- Infinite Integrand

When we have definite integrals of the form

$$\int_a^\infty f(x) dx \text{ and } \int_{-\infty}^b f(x) dx$$

With one limit of integration being infinite, we refer to them as improper integrals.

The first improper integral cited above can be defined to be the limit of another (proper) integral as the latter's upper limit of integration tends to ∞ ; i.e.

$$\int_a^\infty f(x) dx \equiv \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

Example 8.8

Evaluate $\int_1^\infty \frac{dx}{x^2}$

$$x^2$$

we note that

$$\int_1^b \frac{dx}{x^2} = \left[-\frac{1}{x} \right]_1^b = -\frac{1}{b} + 1$$



The desired integral is

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

The improper integral does converge, and it has a value of 1.

By the same token, we can define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

Infinite Integrals

Evaluate $\int_0^1 \frac{1}{x} dx$

$$\int_a^1 \frac{1}{x} dx = \ln x \Big|_a^1 = \ln 1 - \ln a = -\ln a \quad [\text{for } a > 0]$$

And then in the limit as $a \rightarrow 0^+$

$$\int_0^1 \frac{1}{x} dx \equiv \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} (-\ln a)$$

Since this limit does not exist [as $a \rightarrow 0^+$, $\ln a \rightarrow -\infty$] the given integral is divergent.

Some Economic Applications of Integrals

(1) We can obtain marginal function from a given total function through the process of differentiation. Because integration is the converse of differentiation we should be able to infer the total function from a given marginal function.

Example

If the marginal propensity to save (MPS) is the following function of income $S^1(y) = 0.3 - 0.1y^{-1/2}$, and if the aggregate savings S is nil when income Y is 81, find the saving function $S(Y)$.

Solution

$$S(Y) = \int (0.3 - 0.1y^{-1/2}) dy = 0.3y - 0.2y^{1/2} + c$$

The precondition given $S = 0$, when $Y = 81$

Hence,

$$0 = 0.3(81) - 0.2(81)^{1/2} + c$$

$$C = -22.5$$



The desired saving function is

$$S(Y) = 0.3Y - 0.2Y^{1/2} - 22.5.$$

(2) The capital formation is the process of adding to a given stock of capital. If this process is continuous over time, we may express capital stock as a function of time, $K(t)$, and use the derivative dk/dt to denote the rate of capital formation. But the rate of capital formation at time t , is identical with the rate of net investment flow at time t , denoted by $I(t)$. Thus, capital stock K and net investment I are related by the following two equations:

$$\frac{dk}{dt} = I(t)$$

dt

$$\text{and } k(t) = \int I(t) dt = \int \frac{dk}{dt} dt = \int dk$$

dt

Example

Suppose that the net investment flow is described by the equation $I(t) = 3t^{1/2}$ and that the initial capital stock, at time $t = 0$ is $K(0)$. What is the time path of capital K ? by integrating $I(t)$ w.r.t. t , are obtained.

$$k(t) = \int I(t) dt = \int 3(t)^{1/2} dt = 2t^{3/2} + c$$

Next letting $t = 0$, in the leftmost and rightmost expressions, we find $K(0) = c$. Therefore, the time path of K is

$$K(t) = 2t^{3/2} + K(0).$$

Study session summary



Summary

In this study session you learned, about the integral of a function. You went further to apply the concept of integration in Economics.



Assessment



Assessment

SAQ 8.1 (tests Learning Outcome 8.1)

What is the main difference between definite integral and indefinite integral?

SAQ 8.2 (tests Learning Outcome 8.2)

Using integration by part evaluate $\int 2x/(x-8)^3 dx$

SAQ 8.3 (tests Learning Outcome 8.3)

Given $MC = dTC/dQ = 32 + 18Q - 12Q^2$, $FC = 43$. Find the TC, AC and VC functions.



Answers to Self Assessment Questions

SAQ 1.1

Three advantages of using mathematics in economics are:

1. It helps to convey economic information in precise and concise language.
2. It ensures that all necessary assumptions underlying any economic analysis are well stated ab-initio.
3. It helps to provide a wealth of theorems that are useful in economic analysis

Two disadvantages of using mathematics in economics are:

1. It is abstract and a little technical.
2. It could lead to loss of important verbal economic information

SAQ 1.2

Mathematical economics (using graphs and equations)

SAQ 1.3

While econometrics deals with the measurement of economic data using statistics and hypothesis for testing empirical analysis, mathematical economics is concerned with application of mathematics to purely theoretical economics.

SAQ 2.1

A. They cannot be used as ratios, so they can't be used in solving ratios, rates, percentages and fractions related problems in mathematical modelling.

B. Moreover, using them in mathematics require technical knowledge of their functions.

C. They are special numbers that are not needed in all mathematical solutions. For example, $\pi = 3.1427\dots$ is only used in geometry related problems and hardly used in economics.

SAQ 2.2

Let Economics, Sociology and Geography denote E, S and G.

$$\mu = 260, n[E \cup S \cup G] = 240, n[E] = 70, n[S] = 130, n[G] = 150, n[E \cap S] = 50, n[E \cap G] = x, n[S \cap G] = 105, n[E \cap S \cap G] = 40.$$

$$n[E \cup S \cup G] = n[E] + n[S] + n[G] - n[E \cap S] - n[S \cap G] - n[E \cap G] + n[E \cap S \cap G]$$



$$240 = 70 + 130 + 150 - 50 - 105 - x + 40$$

$$240 = 390 - 155 - x$$

$$240 = 235 - x ; x = 240 - 235$$

X = 5, that is, 5 students are offering Economics and Geography

SAQ 2.3

Q can take any value from 0 to 150, so the domain = $\{Q/0 \leq Q \leq 150\}$

The range; Minimum value for C (when $Q = 0$) = $300 + 8(0) = 300$

Maximum value for C (when $Q = 150$) = $300 + 8(150) = 1200$

The range = $\{C/300 \leq C \leq 1200\}$

SAQ 3.1

One linear function is $y = \beta_0 + \beta_1 x$

One cubic function is $y = \alpha_0 - \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ [change of parameters in both functions do not matter]

SAQ 3.2

Non algebraic functions are functions that their independent variables are not expressed in an algebraic way. For example, logarithmic functions like $y = a \log x$ and exponential functions like $y = e^x$

Constant functions are a type of polynomial function which have a range that consists of just one element. They are functions whose highest degree of independent variable is 0. Since any variable raised to 0 equals 1, the independent variable does not appear in the function expressed.

E.g. $y = 15$, $Q = 2$, $y = a_0 \rightarrow x^0$

SAQ 4.1

a.
$$\begin{pmatrix} 4 & 0 \\ 3 & 1 \\ 5 & -1 \end{pmatrix}$$

b.
$$\begin{pmatrix} -4 & 8 & 0 \\ 1 & 5 & 6 \\ 4 & 8 & 2 \end{pmatrix}$$

c.
$$\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$

d.
$$\begin{pmatrix} 1 & 0 & 8 & 4 \end{pmatrix}$$



$$\text{e. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{f. } \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$$

SAQ 4.2

$$\text{a. } \begin{pmatrix} -7 & 7 & 10 \\ 8 & 7 & -5 \end{pmatrix} \qquad \text{b. } \begin{pmatrix} 3 & 3 & -2 \\ -2 & -7 & -7 \end{pmatrix}$$

$$\text{c. } 3A = 3 \begin{pmatrix} -2 & 5 & 4 \\ 3 & 0 & -6 \end{pmatrix} = \begin{pmatrix} -6 & 15 & 12 \\ 9 & 0 & -18 \end{pmatrix}$$

d. For two matrices to be conformable for multiplication, the number of columns in the first must equal the number of rows in the second matrix. Matrix A has 3 columns. However, matrix B has 2 rows. It needs one more row, for the matrix AB to be gotten.

$$\begin{aligned} \text{e. } AC &= \begin{pmatrix} -2(3) + 5(1) + 4(6) \\ 3(3) + 0(1) + -6(6) \end{pmatrix} = \begin{pmatrix} -6 + 5 + 24 \\ 9 + 0 - 36 \end{pmatrix} \\ &= \begin{pmatrix} 23 \\ -27 \end{pmatrix} \end{aligned}$$

Note that since A is a 2×3 matrix and C is a 3×1 matrix, they are matrix conformable for multiplication and their result must be a 2×1 matrix (rows of A and columns of C).

SAQ 5.1

$$\begin{pmatrix} 5 & 3 & 1 \\ -1 & 7 & -2 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 12 \\ 12 \end{pmatrix}$$

SAQ 5.2

$$A^1 = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 5 & 2 \\ 0 & 1 \end{vmatrix} = 5(1) - 2(0) = 5$$



$$B^1 = \begin{pmatrix} 7 & 0 & 2 \\ 6 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$|B| = \begin{vmatrix} 7 & 6 & 2 \\ 0 & 3 & 1 \\ 2 & 0 & 4 \end{vmatrix}$$

$$\begin{aligned} &= |B| = 7 \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ &= 7\{3(4) - 0(1)\} - 6\{0(4) - 2(1)\} + 2\{0(0) - 3(2)\} \\ &= 7(12) - 6(-2) + (-6) \\ &= 84 + 12 - 6 \\ &= 90 \end{aligned}$$

SAQ 5.3

Two proofs exist. First, it is obvious that the elements on the second row are 1.5 times the respective elements on the first row.

$$\begin{bmatrix} 3 & 6 \end{bmatrix} = 1.5 \begin{bmatrix} 2 & 4 \end{bmatrix}$$

Conversely, the elements on the first column are $\frac{1}{2}$ times the respective elements on the second column.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

This means there is a linear interdependence between the first row and the second row and between the first column and the second column. Based on this, matrix A is a singular matrix

Second, we can find the determinant of matrix A and see if our result will equal 0.

$$|A| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 2(6) - 4(3) = 12 - 12 = 0$$

The result shows that matrix A is a singular matrix and as such the inverse of matrix A, A^{-1} does not exist, neither does a unique solution exist for it.

**SAQ 6.1**

$$a. |A| = 7(3) - 6(0) = 21$$

Minor of 7 = 3

Minor of 6 = 0

Minor of 0 = 6

Minor of 3 = 7

$$C = \begin{pmatrix} 3 & 0 \\ 6 & -7 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 3 & 6 \\ 0 & -7 \end{pmatrix} = \text{Adjoint } A$$

$$A^{-1} = \text{Adjoint } A / |A| = 1/21 \begin{pmatrix} 3 & 6 \\ 0 & -7 \end{pmatrix}$$

b. In matrix form,

$$\begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} \quad |A| = 3(1) - (-2)(2) \\ = 3 + 4 \\ = 7$$

To solve for x_1 and x_2 ;

Substituting d into the first column of A to form matrix A_1 and dividing the determinant of A_1 by the determinant of A , we have:

$$A_1 = \begin{pmatrix} 11 & -2 \\ 12 & 1 \end{pmatrix} \quad |A_1| = 11(1) - (-2)(12) \\ = 11 + 24 \\ = 35$$

$$x_1 = |A_1| / |A| = 35/7 = 5$$

Following the same process for the second column of A to solve for x_2

$$A_2 = \begin{pmatrix} 3 & 11 \\ 2 & 12 \end{pmatrix} \quad |A_2| = 3(12) - 2(11) \\ = 36 - 22 \\ = 14$$

$$x_2 = |A_2| / |A| = 14/7 = 2.$$

$$x_1 = 5 \text{ and } x_2 = 2$$



SAQ 6.2

Relevant economic models include input output models, constrained optimization models, general equilibrium models and demand and supply models.

SAQ 7.1

The derivative of a function is the limit of the difference quotient as the independent variable tends towards zero. It measures the rate of change in the dependent variable of a function as the independent variable changes and particularly tends towards zero. If there is a function, $y = f(x)$, as x changes from an initial value x_0 to a new value $(x_0 + \Delta x)$, the value of the function $y = f(x)$ changes to $f(x_0 + \Delta x)$. This change is represented by the difference quotient derived as;

$$y + \Delta y = f(x_0 + \Delta x)$$

$$\Delta y = f(x_0 + \Delta x) - y$$

$$\Delta y = f(x_0 + \Delta x) - f(x)$$

$$\Delta y = f(x_0 + \Delta x) - f(x)$$

$$\Delta x \Delta x$$

SAQ 7.2

a. $y = 7x^2 + 3$, using power rule;

$$dy/dx = 14x$$

b. $y = (3x + 2)(4x^2)$, using product rule,

$$dy/dx = [(3x + 2)8x + (4x^2)3]$$

$$dy/dx = [24x^2 + 16x + 12x^2]$$

$$dy/dx = [36x^2 + 16x] = 4x[9x + 4]$$

c. $y = (4x^2 - 7x)/3x^2$, using quotient rule;

$$dy/dx = [(3x^2)(8x - 7) - (4x^2 - 7x)6x] / (3x^2)^2$$

$$dy/dx = [24x^3 - 21x^2 - 24x^3 + 42x^2] / 9x^4$$

$$dy/dx = 21x^2 / 9x^4 = 7x / 3x^2$$

SAQ 7.3

a. Given $TR = 900q - 50q^2$

$$TC = 420 + 210q - 60q^2 + 75q^3$$

Using $y = h(x) - g(x)$, where $dy/dx = dh(x)/dx - dg(x)/dx$

$$\Pi = TR - TC$$

$$d\Pi/dq = dTR/dq - dTC/dq$$

$$d\Pi/dq = (900 - 100q) - (210 - 120q + 225q^2)$$



$$\begin{aligned}
 &= 900 - 100q - 210 + 120q - 225q^2 \\
 &= 690 + 20q - 225q^2 = 0 \\
 &= -225q^2 + 404q - 384q + 690 \\
 &= (-q - 1.7)(225q - 404) = 0 \\
 &-q - 1.7 = 0 ; q = -1.7 \\
 &225q - 404 = 0 ; q = 1.8
 \end{aligned}$$

Using the positive value, $q = 1.8$

$$\text{b. } \Pi = \text{TR} - \text{TC}$$

$$\Pi = 900q - 50q^2 - [420 + 210q - 60q^2 + 75q^3]$$

Opening the brackets and substituting $q = 1.8$ in, we have

$$\Pi = -75(1.8)^3 - 10(1.8)^2 + 690(1.8) - 420$$

$$\Pi = 417$$

SAQ 8.1

The definite integral is a unique anti-derivative or integral which can be evaluated by using the fundamental theorem of calculus/ The fundamental theorem of calculus states that the numerical value of the definite integral of a continuous function $f(x)$ over the interval from a to b is given by the indefinite integral evaluated at the upper limit b minus the same indefinite integral evaluated at the lower limit of integration, a , while the constants which are common to both evaluations are eliminated through subtraction.

$$\int_a^b f(x) dx = F(x) \quad \int_a^b = F(b) - F(a)$$

On the other hand, indefinite integral is a set of all possible integrals or antiderivative of the integrand $f(x)$.

SAQ 8.2

$$\int 2x/(x-8)^3 dx$$

Let $f(x) = 2x$, $f'(x) = 2$, and $g'(x) = (x-8)^{-3}$, then $g(x) = \int (x-8)^{-3} dx = -1/2(x-8)^{-2}$

Given the formula for integration by part;

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

$$\begin{aligned}
 \int 2x/(x-8)^3 dx &= 2x \{-1/2(x-8)^{-2}\} - \int -1/2(x-8)^{-2} 2 dx \\
 &= -x(x-8)^{-2} + \int (x-8)^{-2} dx \\
 &= -x(x-8)^{-2} - (x-8)^{-1} + c
 \end{aligned}$$

SAQ 8.3

$$\text{TC} = \int \text{MC} dQ = \int (32 + 18Q - 12Q^2) dQ$$



$$= 32Q + 9Q^2 - 4Q^3 + c$$

At $Q = 0$, $TC = FC$. Since $FC = 43$, $c = 43$

$$TC = 32Q + 9Q^2 - 4Q^3 + 43$$

$$AC = TC/Q = 32 - 9Q - 4Q^2 + (43/Q)$$

$$VC = TC - FC = 32Q + 9Q^2 - 4Q^3$$