ECO 204
Introductory Mathematics for Economists II

PROPERTY OF DISTANCE LEARNING CENTRE, UNIVERSITY OF IBADAN

PROPERTY OF DISTANCE LEARNING CENTRE, UNIVERSITY OF IBADAN

Ibadan Distance Learning Centre Series

ECO 204
Introductory Mathematics for Economists II

By
Musbau Adetunji Babatunde, Ph.D
Department of Economics
University of Ibadan

PROPERTY OF DISTANCE LEARNING CENTRE, UNIVERSITY OF IBADAN

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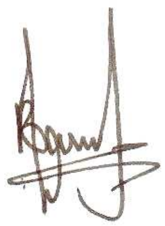
Vice-Chancellor's Message

I congratulate you on being part of the historic evolution of our Centre for External Studies into a Distance Learning Centre. The reinvigorated Centre, is building on a solid tradition of nearly twenty years of service to the Nigerian community in providing higher education to those who had hitherto been unable to benefit from it.

Distance Learning requires an environment in which learners themselves actively participate in constructing their own knowledge. They need to be able to access and interpret existing knowledge and in the process, become autonomous learners.

Consequently, our major goal is to provide full multi media mode of teaching/learning in which you will use not only print but also video, audio and electronic learning materials.

To this end, we have run two intensive workshops to produce a fresh batch of course materials in order to increase substantially the number of texts available to you. The authors made great efforts to include the latest information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly. It is our hope that you will put them to the best use.



Professor Olufemi A. Bamiro, FNSE
Vice-Chancellor

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Foreword

The University of Ibadan Distance Learning Programme has a vision of providing lifelong education for Nigerian citizens who for a variety of reasons have opted for the Distance Learning mode. In this way, it aims at democratizing education by ensuring access and equity.

The U.I. experience in Distance Learning dates back to 1988 when the Centre for External Studies was established to cater mainly for upgrading the knowledge and skills of NCE teachers to a Bachelors degree in Education. Since then, it has gathered considerable experience in preparing and producing course materials for its programmes. The recent expansion of the programme to cover Agriculture and the need to review the existing materials have necessitated an accelerated process of course materials production. To this end, one major workshop was held in December 2006 which have resulted in a substantial increase in the number of course materials. The writing of the courses by a team of experts and rigorous peer review have ensured the maintenance of the University's high standards. The approach is not only to emphasize cognitive knowledge but also skills and humane values which are at the core of education, even in an ICT age.

The materials have had the input of experienced editors and illustrators who have ensured that they are accurate, current and learner friendly. They are specially written with distance learners in mind, since such people can often feel isolated from the community of learners. Adequate supplementary reading materials as well as other information sources are suggested in the course materials.

The Distance Learning Centre also envisages that regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks will find these books very useful. We are therefore delighted to present these new titles to both our Distance Learning students and the University's regular students. We are confident that the books will be an invaluable resource to them.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.

Professor Francis O. Egbokhare
Director

Course Introduction

This course builds on the basic mathematical foundation set out in ECO 104 as well as delving into detailed economic application of those topics.

It also provides the basic requirements for microeconomics, macroeconomics and mathematical economics at the higher levels.

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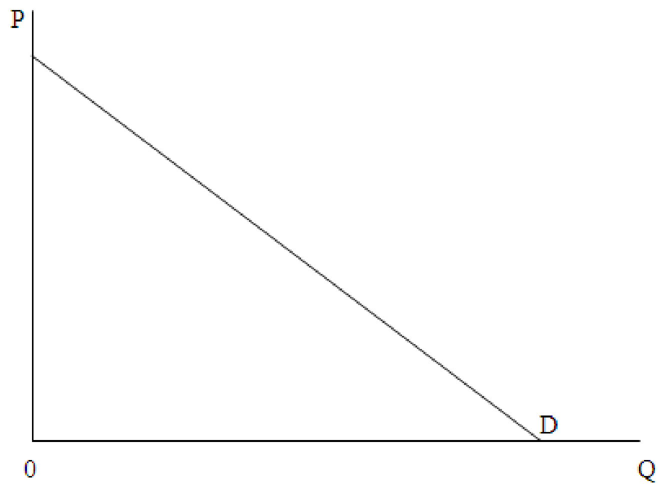
LECTURE ONE

Linear Functions

Linear functions are functions whose graphs are straight line.

An example of such can be Q = 123 – 3P

The two variables present in the function above are Q (quantity) and P (price) are respectively dependent and independent variables. It should be noted that the graph of the above equation slopes downward from left to right because of the negative sign on the slope.



The above equation can be written general form;

$Q = A + BP + CY + D\pi + ET$

Where A, B, C, D and E are constants and also for the certainty of its linearity, all the variables except Q and P must be held constant. However any variation in those other variables will lead to a shift in the curve.

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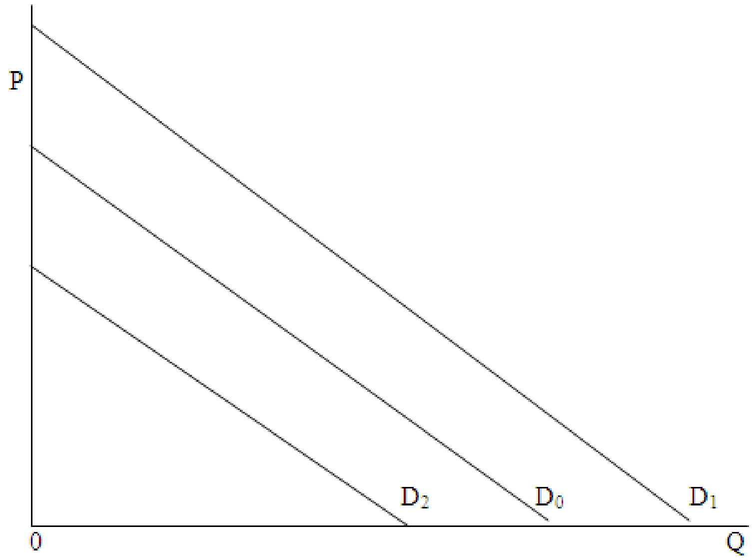


Economics makes use of certain linear functions and these can be shown as in the following:

1. **Tax laws**: Income tax is a fixed proportion of pre-tax income. This can be written in algebra as; $D = Y - tY$

Where D = disposable income

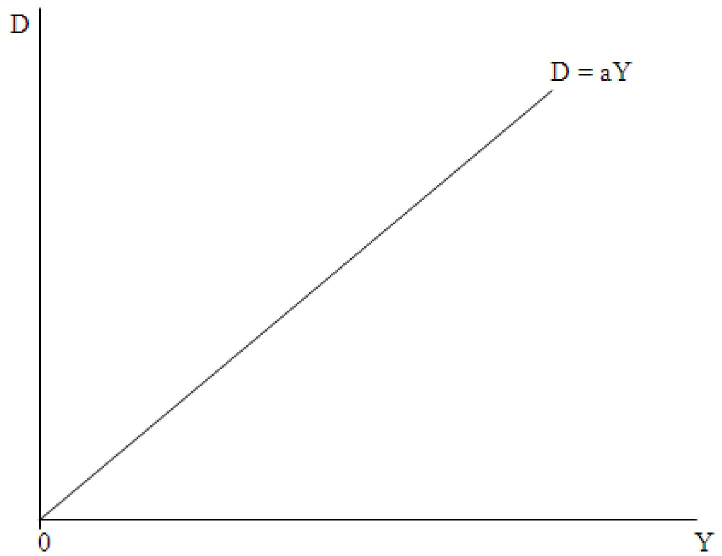
Y = Income

t = fixed tax proportion/rate $0 < t < 1$

then $D = (1 - t)Y$

$D = aY$ where $a = (1-t)$

Note that the function is a positively sloped straight line function because the disposable income increases as income increases. Hence a positive relationship.



2. **Consumption function**: The consumption function relates consumption expenditure to the disposable income.

C = a + bD   where a and b are constant

a = autonomous consumption

b = marginal propensity to consume

C = consumption expenditure (Explained variable)

D = disposable income

Derived relations:

C = a + bD ……….. i

D = (1 – t) Y …………ii

Substituting equation (ii) into (i), we have;

C = a + b[(1 – t) Y]

 = a + (b – bt) Y

If b – bt = d, then

C = a + dY

Now, the above shows consumption as directly related to income before tax.

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3. **Simultaneous linear functions**: A very good example of this type of function is the demand and supply functions.

e.g. consider the demand and supply functions given below

$Q^d = 30 – 2P$

$Q^s = -10 + 3P$

Find the equilibrium price and quantity.

Solution

At equilibrium, the consumer and the producer are expected to have reached an agreement on the quantity that will be bought and sold as well as the price that will be paid and received. Therefore, equating the demand function to the supply function will help solve the problem.

i.e $Q^d = Q^s$

$30 – 2P = -10 + 3P$

$30 + 10 = 3P + 2P$

$40 = 5P$

$P = 40/5$

$P = 8$

Substitute P = 8 into any of demand or supply function to get the equilibrium quantity;

$Q^d = 30 – 2(8) = 30 -16 =14$ OR

$Q^s = -10 + 3(8) = -10 + 24 = 14$

Q = 14 and P = 8

4. Cases in Economics where some arguments lead to linear equations and some statements leading to simultaneous linear equations.

Example: suppose commodities m and n are related in linear form and as shown in the following functions

$Qm = K_1 – a_{11}Pm + a_{12}Pn$

$Qn = K_2 + a_{21}Pm – a_{22}Pn$

If we take the specific form

$Qm = 60 – 6Pm + 4Pn$

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$Qn = 25 + 2Pm – 5Pn$

Suppose that $Qm = 24$ and $Qn = 15$, then

$24 = 60 – 6Pm + 4Pn$ …………i

$15 = 25 + 2Pm – 5Pn$ ………….ii

From equation (i) we have

$6Pm – 4Pn = 60 – 24$

$6Pm – 4Pn = 36$ ……………iii

From equation (ii) we have

$2Pm – 5Pn = 15 – 25$

$2Pm – 5Pn = -10$ …………….iv

Equations (i) and (ii) have formed a simultaneous linear equation in which two variables are related to each other. The prices of the two commodities can now be obtained by solving those equations.

Multiply equation (iv) by 3

$6Pm – 15Pn = -30$ ……………..v

Subtract equation (v) from (iii)

$6Pm – 4Pn – 6Pm + 15Pn = 36 + 30$

$11Pn = 66$

Pn = 6

Substitute Pn = 6 into equation (iii)

$6Pm – 4(6) = 36$

$6Pm – 24 = 36$

$6Pm = 36 + 24$

$6Pm = 60$

$Pm = 60/6$

Pm = 10

Example 2: The quantity demanded of rice is given by the equation

$Q = 60 – 1/3P$

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a. Find the quantity demanded of rice at prices of 0, 15, and 105
b. Express the inverse of the equation
c. Plot your answer on a graph. What type of graph did you derive?

Solution:

$Q = 60 – 1/3P$

a. when price = 0

$Q = 60 – 1/3(0)$

$Q = 60 – 0$

$Q = 60$

when price = 15

$Q = 60 – 1/3(15)$

$Q = 60 – 5$

$Q = 55$

When price = 105

$Q = 60 – 1/3(105)$

$Q = 60 – 35$

$Q = 25$

b. Expressing the inverse of the function means make P the subject of the formula

$Q = 60 – 1/3P$

$1/3P = 60 – Q$

Multiply through by 3

$3 \times {}^{1}/_{3}\,P = (60 – Q) \times 3 = 60 \times 3 – Q \times 3$

$P = 180 – 3Q$

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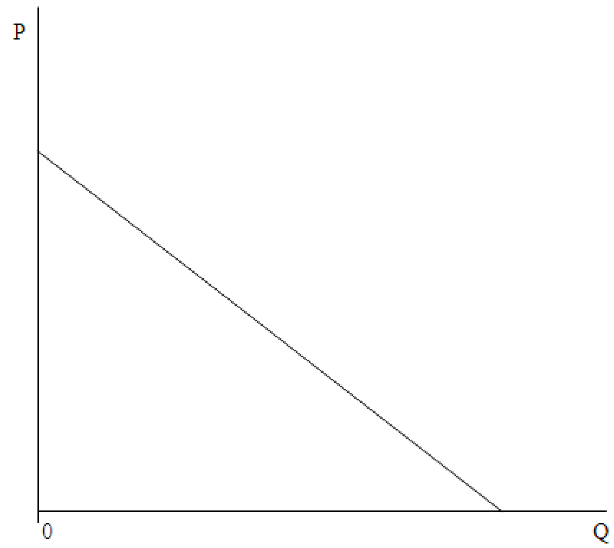

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Post-Test

1. The market for Sugar is characterized by the following demand and supply functions

   $Qd + 2P = 16$ and $Qs + 4 = 3P$

   Determine the equilibrium price and quantity.

2. Determine the equilibrium prices and quantities for the following markets

   a. $Qs = 3P – 20$ and $P = 44 – 0.2Qd$

   b. $P = 0.125Qs – 45$ and $Qd = 125 – 2P$

   c. $Qs + 32 = 7P$ and $Qd – 128 + 9P = 0$

   d. $13P = Qs – 27$ and $Qd + 4P – 24 = 0$

3. A consumer's demand function for petrol is represented as $Q = 30 – 1/3P$ where Q is quantity demanded and P is price per litre.

   a. find the quantity demanded when price is ₦11 and ₦ 22 per litre.

   b. Suppose the function changes to take the form $Q = 20 – 1/3P$, find the quantity demanded at the same prices given in a.

LECTURE TWO

The Firm's Behaviour: Revenue, Cost and Profit Functions

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Total Revenue Function

If a demand function assumes $P = K - aQ$ being a relationship between price and quantity demanded, then

$$\text{Total Revenue (TR)} = P \times Q = (K - aQ)\,Q$$

$$TR = KQ - aQ^2$$

The TR function here is a second degree polynomial and the curve is curvilinear. In general, **linear demand functions give rise to curvilinear TR functions**.

Profit Function

If the firm's Total Cost (TC) function takes the form

$TC = F + bQ$ (where F = Fixed Cost and b = Variable cost per unit)

Then TR – TC gives a profit function in terms of Q i.e f(Q)

If we denote profit by π, we can have

$$\pi = TR - TC = KQ - aQ^2 - (F + bQ)$$

$$\pi = KQ - aQ^2 - F - bQ$$

Which we can further express as

$$\pi = -aQ^2 + (K - b)Q - F$$

The profit function is also a curvilinear function i.e a second degree polynomial.

Given this background, it is possible to find the quantity of goods that should be produced to satisfy the level of profit that the producer may want to attain at any point in time.

Assume that the parameters K, F, a and b take the values 18, 50, 1 and 3 respectively i.e the demand function is P = 18 – Q and the total cost function is TC = 50 + 3Q

$TR = P \times Q = 18Q - Q^2$; $TC = 50 + 3Q$

$\pi = -Q^2 + (18 - 3)Q - 50$

$\pi = -Q^2 + 15Q - 50$

Let us now solve for Q at different profit levels

e.g.

At what level of output will the firm breakeven (i.e. a point when profit = 0)

If $\pi = 0$, then

$-Q^2 + 15Q - 50 = 0$

$Q^2 - 15Q + 50 = 0$

Factorize to solve for Q

$(Q - 5)(Q - 10) = 0$

$Q - 5 = 0; \quad Q = 5$

OR $Q - 10 = 0; \quad Q = 10$

It is also possible to find the level of out put (Q) at which profit is equal to say 6.

If $\pi = 6$, then

$-Q^2 + 15Q - 50 = 6$

$Q^2 - 15Q + 50 = -6$

$Q^2 - 15Q + 50 + 6 = 0$

$Q^2 - 15Q + 56 = 0$

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Factorizing the equation, we have

$(Q – 7) (Q – 8) = 0$

$Q = 7$ or $Q = 8$

Example 2: Suppose the demand function of a firm is given by $Q + P – 20 = 0$ and its cost $TC – 48 – 4Q = 0$. Find the largest Q it can produce consistently with

a. breaking even
b. making a profit of ₦12
c. making a loss of ₦20

Solution:

$P = 20 – Q$

Thus, $TR = P \times Q = 20Q – Q^2$

$TC = 4Q + 48$

Profit function $\pi = TR – TC = 20Q – Q^2 – 4Q – 48$

$\Pi = -Q^2 + 16Q – 48$

a. $\pi = 0$

$-Q^2 + 16Q – 48 = 0$

$Q^2 – 16Q + 48 = 0$

$(Q – 4) (Q – 12) = 0$

The largest quantity for breaking even is $Q = 12$

b. $\pi = 12$

$-Q^2 + 16Q – 48 = 12$

$Q^2 – 16Q + 48 = -12$

$Q^2 – 16Q + 48 + 12 = 0$

$Q^2 – 16Q + 60 = 0$

$(Q – 6) (Q – 10) = 0$

$Q = 6$ or $Q = 10$

The largest quantity for making a profit of ₦12 is $Q = 10$

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c. $\pi = -20$

$-Q^2 + 16Q - 48 = -20$

$Q^2 - 16Q + 48 = 20$

$Q^2 - 16Q + 48 - 20 = 0$

$Q^2 - 16Q + 28 = 0$

$(Q - 2)(Q - 14) = 0$

$Q = 2$ or $Q = 14$

The largest level of output consistent with making a loss of ₦20 is $Q = 14$

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LECTURE THREE

Curvilinear Demand and Supply Functions

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Example 1:

Suppose the market supply function is $Ps = Q^2 + 4Q + 1$ and the market demand function takes the form $Pd = -Q^2 + Q + 4$, find the equilibrium price and quantity

Solution:

At equilibrium, demand must be equal to supply

$Q^2 + 4Q + 1 = -Q^2 – Q + 4$

$2Q^2 + 5Q – 3 = 0$

$(2Q – 1)(Q + 3) = 0$

$Q = ½$ or $Q = - 3$

Negative quantity does not make economic sense, therefore the quantity is **Q = ½.**

Substitute $Q = ½$ into any of supply or demand function to get the price

$P = Q^2 + 4Q + 1$

$(1/2)^2 + 4(1/2) + 1$

$¼ + 2 + 1$

P = 3.25

Example 2:

Given the market demand function $P + Q^2 + Q = 11$ and the market supply function

$2P - 2Q^2 + Q - 4 = 0$ Calculate the equilibrium price and quantity in the market.

Solution:

From the demand function,

$P = 11 - Q - Q^2$

And from the supply function,

$P = Q^2 - 0.5Q + 2$

At equilibrium, demand = supply

$11 - Q - Q^2 = Q^2 - 0.5Q + 2$

$-2Q^2 - 0.5Q + 9 = 0$

Multiply through by -2

$4Q^2 + Q - 9 = 0$

$(Q - 2)(4Q + 9) = 0$

$Q - 2 = 0$ 0r $4Q + 9 = 0$

Q = 2

The other quantity is negative and it is therefore neglected.

Average Cost Functions

Average costs are costs per unit of output. To obtain average cost, total cost is divided by quantity produced.

i.e. AC = TC/Q

since TC = F + bQ, then

$$AC = \frac{F + bQ}{Q}$$

$$= \frac{F}{Q} + b$$

If the Fixed cost (F) is 40 and b = 3,

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Then TC = 40 + 3b

And AC = $\frac{40}{Q}$ + 3

GRAPH 5

A hyperbola always has two branches. The function above $\frac{40}{Q} + 3$ represents a type of hyperbola called equilateral or rectangular hyperbola. A rectangular hyperbola is one whose asymptotes are perpendicular to each other and parallel to the coordinate axis. This type of hyperbola takes the form (x – u) (y – m) = r

Where u, m and r are constant and x and y are the variables.

x = u and y = m are the asymptotes of the rectangular hyperbolic function.

To see whether we can reduce the Average cost function to this standard form

$$Z = \frac{40}{Q} + 3$$

ZQ = 40 + 3Q

ZQ – 3Q = 40

(Q – 0) (Z – 3) = 40

Here, Q = 0 and Z = 3 are the asymptotes of the hyperbolic function, and Q and Z form the variables.

Exercise:

Suppose a firm producing cassava chips has the following cost and revenue functions

TC = 500 + 2Q; TR = 8Q

i. identify the fixed cost and the variable cost from the equations
ii. what proportion of the total revenue is the variable cost
iii. at what quantity is profit equal to zero
iv. would you say the firm is in perfect competition or an imperfect competition (Hint: use the nature of the TR)

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LECTURE FOUR

The IS – LM Model

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The IS – LM model describes the overall equilibrium position in an economy. It is described as overall because both the goods/commodity and money markets are in equilibrium at the same time.

The goods market equilibrium requires that the demand for goods be equal to the supply of goods. For a closed economy, the IS equation is represented by

$Y = C + I + G$

Where Y = equilibrium income

C = consumption expenditure

I = investment

G = government spending

Further defined,

$C = a + bY$

Where a = autonomous consumption

b = marginal propensity to consume

and $I = \alpha - \beta i$

where α = autonomous investment

β = part of investment that depends on interest rate

i = interest rate

G^* = autonomous government spending.

Similarly, equilibrium in the money market requires that the demand for money be equal to its supply. Thus;

$M_d = M_s$

At any point in time, the money supply in an economy is constant (fixed) as it is determined by the monetary authority (Central Bank).

However, money demand is a function of income and interest rate. It is positively related to income (transaction balance) and negatively related to interest rate (speculative balance).

$M_d = qY + K - \lambda i$

With the two equilibrium positions, it is possible to examine the overall equilibrium income and interest rate in the economy.

Example 1: Assume a two sector economy where C = 48 + 0.8Y, I = 98 – 75i, Ms = 250 and Md = 0.3Y + 52 – 150i

a. solve for the equilibrium Y and i
b. at the equilibrium, what is the level of consumption and investment

Solution

a. Commodity market equilibrium exists when Y = C + I

Y = 48 + 0.8Y + 98 – 75i

Y – 0.8Y = 146 – 75i

0.2Y + 75i – 146 = 0 …………………..i

Monetary equilibrium exists when Ms = Md

250 = 0.3Y + 52 – 150i

0.3Y = 250 – 52 + 150i

0.3Y = 198 + 150i

0.3Y – 150i – 198 = 0 ……………………ii

Solve equations (i) and (ii) simultaneously

0.2Y + 75i – 146 = 0 ………………….i

0.3Y – 150i – 198 = 0 …………………ii

Multiply equation (i) by 2

0.4Y + 150i – 292 = 0 …………………iii

Add equations (ii) and (iii) together to eliminate i

0.7Y – 490 = 0

0.7Y = 490

Y = 700

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Substitute Y = 700 into either equation (i) or (ii) to find i

$0.2Y + 75i – 146 = 0$

$0.2(700) + 75i – 146 = 0$

$140 + 75i -146 = 0$

$75i – 6 = 0$

$75i = 6$

i = 0.08

both commodity and money markets are in equilibrium when Y = 700 and i = 0.08

b. to obtain equilibrium C, substitute Y = 700 into consumption function

$C = 48 + 0.8Y$

$48 + 0.8(700)$

$48 + 560$

C = 608

For I, I = 98 – 75i

$98 – 75(0.08)$

$98 – 6$

I = 92

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Example 2: Given C = 102 + 0.7Y; I = 150 – 100i; Ms = 300 and Md = 0.25Y + 124 – 200i. Find

a. the equilibrium level of income and the equilibrium rate of interest
b. the level of C and I when the economy is in equilibrium
c. if government spending G is introduced into the model, how is the equilibrium Y and I affected.

Solution:

Commodity market equilibrium (IS) exists when

$Y = C + I$

$Y = 102 + 0.7Y + 150 – 100i$

$Y – 0.7Y = 252 – 100i$

$0.3Y = 252 – 100i$

$0.3Y + 100i – 252 = 0$ ………………i

Monetary equilibrium (LM) exists when

$Ms = Md$

$300 = 0.25Y + 124 – 200i$

$0.25Y – 200i – 176 = 0$ …………….ii

Then solve eqns (i) and (ii) simultaneously

$0.3Y + 100i -252 = 0$ …………………i

$0.25Y – 200i – 176 = 0$ ……………….ii

Multiply equation (i) by 2 to eliminate i

$0.6Y + 200i – 504 = 0$ …………………iii

Add equations (ii) and (iii) together

$0.85Y – 680 = 0$

$0.85Y = 680$

Y = 800

Substitute Y = 800 into eqn (i) or (ii)

$0.25Y – 200i – 176 = 0$

$0.25(800) – 200i – 176 = 0$

$200 – 200i – 176 = 0$

$-200i + 24 = 0$

$200i = 24$

i = 0.12

b. at Y = 800 and i = 0.12

$C = 102 + 0.7Y$

$102 + 0.7(800)$

$102 + 560$

C = 662

$I = 150 – 100i$

$150 – 100(0.12)$

$150 – 12$

I = 138

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c. the introduction of government spending will affect the IS equation, thus

$Y = C + I + G$

$Y = 102 + 0.7Y + 150 – 100i + 100$

$Y – 0.7Y = 352 – 100i$

$0.3Y = 352 – 100i$

$0.3Y + 100i – 352 = 0$ ………………….iv

The monetary equilibrium is not affected, so we retain eqn (ii) and have

$0.3Y + 100i – 352 = 0$ ………………….iv

$0.25Y – 200i – 176 = 0$ …………………ii

Multiply eqn (iv) by 2 to eliminate i

$0.6Y + 200i -704 = 0$ ……………………..v

Add equations (v) and (ii) together

$0.85Y – 880 = 0$

$0.85Y = 880$

$=880/0.085$

Y = 1035.3

Substitute Y = 1035.3 into eqn (iv) or (ii) to find i

$0.25Y – 200i – 176 = 0$

$0.25(1035.3) – 200i – 176 = 0$

$258.83 – 200i – 176 = 0$

$82.83 – 200i = 0$

$200i = 82.83$

i = 0.4

Exercise:

Find (a) the equilibrium income level and interest rate at the levels of C and I in equilibrium when

$C = 89 + 0.6Y$; $I = 120 – 150i$; $Ms = 275$ and $Md = 0.1Y + 240 – 250i$

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LECTURE FIVE

Differentiation

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Introduction

The process of finding derivative is known as differentiation. A set of rules of differentiation exist for finding the derivatives of many common functions. Although there are many functions for which the derivative does not exist, our concern will be with functions which are differentiable.

Differentiation in calculus, denoted by $\frac{dy}{dx}$ is equivalent to the slope of a curve in geometry.

$\frac{dy}{dx}$ explains how an infinitesimal change in x brings about a change in y. An alternative to the $\frac{dy}{dx}$ notation is the $f^{'}$(x) (read "**f prime of x**").

Rules of Differentiation

The rules presented in this section have been developed using the limit approach. For our purpose it will suffice to present the rules without proof. The rules of differentiation apply to functions which have specific structural characteristics/forms.

Rule 1: Constant function Rule.

The derivative of a constant function y = f(x) = k is equal to zero for all values of x.

i.e $\frac{dy}{dx}$ = $f^{'}$(x) given that x is a constant

As we pointed out earlier, the derivative of a function has its Geometric counterpart in the slope of the curve. The graph of the constant function is a horizontal straight line with a zero slope. i.e. its rate of change is zero.

This can be applied to fixed cost (FC) in Economics.

Given that y = FC = f(Q) = 2000

$$\frac{dy}{dQ} = \frac{d(2000)}{dQ} = 0$$

GRAPH 6

The implication is that fixed cost, or any other constant function, does not change in response to changes in quantity.

Rule 2: Power function Rule

The derivative of a power function $y = f(x) = x^n$ is equal to nx^{n-1}

To carry out this operation, you use the power of x to multiply the function and then subtract 1 (one) from the original power of x in the given function to get the power of your derivative.

e.g Find the derivative of $y = x^5$

$$\frac{dy}{dx} = 5x^{5-1} \quad = 5x^4$$

e.g $y = f(x) = x^8$

$$\frac{dy}{dx} = 8x^{8-1} = 8x^7$$

Note that such function as above may also be given with a constant known as the coefficient of x. for example, $y = f(x) = Cx^n$ and the derivative becomes

$$\frac{dy}{dx} = Cnx^{n-1}$$

e.g y = f(x) = 4x

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$\frac{dy}{dx}$ = $4(1)x^{1-1}$

$4x^0$

Since $x^0 = 1$

Then $\frac{dy}{dx} = 4$

Given that $y = 6x^{-2}$ find the derivative of y with respect to x

$\frac{dy}{dx} = 6(-2)x^{-2-1}$

$= -12x^{-3}$

If $y = -3x^2$, find $\frac{dy}{dx}$

$\frac{dy}{dx} = -3(2)x^{2-1}$

$= -6x$

e.g $y = 2x^{1/2}$

$\frac{dy}{dx} = 2(1/2)\ x^{1/2-1}$

$= x^{-1/2}$

e.g Given that $y = 3/x^2$, find $\frac{dy}{dx}$

Note that the above function can be rewritten as $3x^{-2}$, now differentiating becomes a simple task

$\frac{dy}{dx} = 3(-2)x^{-2-1}$

$= -6x^{-3}$

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Rule 3: Sum or Difference Rule

This rule implies that the derivative of a function formed by the sum (difference) of two or more component functions is the sum (difference) of the derivatives of the component functions.

$$\frac{d[f(x) \pm g(x)]}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$

Example 1: Consider the function y = f(x) = 5x4 – 8x3 + 3x2 – x + 50. The derivative is found by differentiating each term in the function. Thus,

$$\frac{dy}{dx} = 5(4x^{4-1}) - 8(3x^{3-1}) + 3(2x^{2-1}) - x^{1-1} + 0$$

$$= 5(4x^3) - 8(3x^2) + 3(2x) - x^0$$

$$\frac{dy}{dx} = 20x^3 - 24x^2 + 6x - 1$$

Example 2: Find $\frac{dy}{dx}$ if

$$f(x) = 8x^3 - 4x^2 + 3x - 10$$

$$\frac{dy}{dx} = 8(3x^{3-1}) - 4(2x^{2-1}) + 3(x^{1-1}) - 0$$

$$8(3x^2) - 4(2x) + 3(x^0)$$

$$\frac{dy}{dx} = 24x^2 - 8x + 3$$

Rule 4: Product Rule

The derivative of the product of two functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function.

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= f(x)g'(x) + g(x)f'(x)$$

It is conventional to have it written as

$$\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx}$$

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In which case the function to be differentiated is written as

f(x) = U(x) . V(x)

Example 1: find $\frac{dy}{dx}$ $if \ f(x) = (x^2 - 5)(x - x^3$

Solution:

U(x) = $x^2 - 5$

V(x) = $x - x^3$

Applying the formula $\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx}$, we have

$(x^2 - 5)(1 - 3x^2) + (x - x^3)2x$

$x^2 - 3x^4 - 5 + 15x^2 + 2x^2 - 2x^4$

$\frac{dy}{dx} = -15x^4 + 18x^2 - 5$

Example 2: find $\frac{dy}{dx}$ if f(x) = $(x^3 - 2x^5)(x^4 - 3x^2 + 10)$

Solution:

$U(x) = x^3 - 2x^5$

$V(x) = x^4 - 3x^2 + 10$

$\frac{du}{dx} = 3x^2 - 10x^4$

$\frac{dv}{dx} = 4x^3 - 6x$

$\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx}$

$(x^3 - 2x^5)(4x^3 - 6x) + x^4 - 3x^2 + 10)(3x^2 - 10x^4)$

Expand the brackets

$4x^6 - 6x^4 - 8x^8 + 12x^6 + 3x^6 - 10x^8 - 9x^4 + 30x^6 + 30x^2 - 100x^4$

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Collect like terms

$$-8x^8 - 10x^8 + 4x^6 + 12x^6 + 3x^6 + 30x^6 - 6x^4 - 9x^4 - 100x^4 + 30x^2$$

$$= -18x^8 + 49x^6 - 115x^4 + 30x^2$$

Rule 5: Quotient Rule

If $f(x) = \frac{U(x)}{V(x)}$ where U and V are differentiable and $V(x) \neq 0$, then

$$f'(x) = \frac{V(x).U'(x) - U(x).V'(x)}{V^2}$$

OR

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

Note: unlike in the product rule where we have a sum of two products, the quotient rule involves the difference between them. The implication of this is that while the arrangement does not matter in product rule (addition is commutative), it matters in quotient rule.

i.e. for product rule, $U\frac{dv}{dx} + V\frac{du}{dx} = V\frac{du}{dx} + U\frac{dv}{dx}$

but for quotient rule, $\frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2} \neq \frac{U\frac{dv}{dx} - V\frac{du}{dx}}{V^2}$ the one on the right is wrong

Example 1: Consider f(x) = $\frac{(3x^2 - 5)}{(1 - x^3)}$ applying quotient rule with

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$U(x) = 3x^2 - 5, \frac{du}{dx} = 6x$

$V(x) = 1 - x^3, \frac{dv}{dx} = -3x^2$

$f'(x) = \frac{dy}{dx} = \frac{(1-x^3)6x - (3x^2-5)(-3x^2)}{(1-x^3)^2}$

$\frac{6x - 6x^4 + 9x^4 - 15x^2}{(1-x^3)^2}$

$= \frac{3x^4 - 15x^2 + 6x}{(1-x^3)^2}$

Example 2: If $f(x) = \frac{(-x^3+1)}{(x^5-20)}$, find $f'(x)$

Solution:

$$f'(x) = \frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

$U(x) = -x^3 + 1, \frac{du}{dx} = -3x^2$

$V(x) = x^5 - 20, \frac{dv}{dx} = 5x^4$

Where $\frac{dy}{dx}$ $\frac{(x^5-20)(-3x^2) - (-x^3+1)(5x^4)}{(x^5-20)^2}$

$\frac{-3x^7 + 60x^2 + x^3 - 1 - 5x^4}{(x^5-20)^2}$

$= \frac{-3x^7 - 5x^4 + x^3 + 60x^2 - 1}{(x^5-20)^2}$

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Rule 6: Power of a Function

If $f(x) = [U(x)]^n$ where U is a differentiable function and n is a real number, then

$f'(x) = n.[U(x)]^{n-1}.U'(x)$

Example 1: consider $f(x) = \sqrt{7x^4} - 5x - 9,\ find\ f(x)$

Solution:

Note that the function can be rewritten as

$f(x) = (7x^4 - 5x - 9)^{1/2}$

We can apply rule 6, where $U(x) = 7x^4 - 5x - 9$

$$f'(x) = n[U(x)]^{n-1}.U'(x)$$

$$= \tfrac{1}{2}(7x^4 - 5x - 9)^{1/2-1}(28x^3 - 5)$$

$$= \tfrac{1}{2}(7x^4 5x - 9)^{-1/2}(28x^3 - 5)$$

$$= \tfrac{1}{2}(28x^3 - 5)(7x^4 - 5x - 9)^{-1/2}$$

$$f'(x) = (14x^3 - \tfrac{5}{2})(7x^4 - 5x - 9)^{-1/2}$$

Example 2: Find the derivative of $\left(\dfrac{3x}{1 - x^2}\right)^5$

This function has the form stated in rule 6, where U is the rational function $\dfrac{3x}{(1 - x^2)}$

[note that apart from applying rule 6, you must also apply quotient rule to find $u'(x)$]

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$f'(x) = n[U(x)]^{n-1}.U'(x)$

$= 5\left(\frac{3x}{1-x^2}\right)^4 \frac{(1-x^2)(3)-(3x)(-2x)}{(1-x^2)^2}$

$= 5\left(\frac{3x}{1-x^2}\right)^4 \frac{3-3x^2+6x^2}{(1-x^2)^2}$

$= 5\left(\frac{3x}{1-x^2}\right)^4 \frac{3+3x^2}{(1-x^2)^2}$

Rule 7: Chain Rule

If we have a function y = f(u), where u is in turn a function of another variable x, say U = g(x), then the derivative of y with respect to x is equal to the derivative of y with respect to u, times the derivative of u with respect to x.

Expressed as $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Example 1: Given that $y = f(u) = u^2 - 2u + 1$ and $u = g(x) = x^2 - 1$

Then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = (2u-2)(2x)$

We then need to substitute $(x^2 - 1)$ for u so that $\frac{dy}{dx}$ strictly becomes a function of x.

$\frac{dy}{dx} = [2(x^2-1)-2](2x)$

$= (2x^2 - 2 - 2)(2x)$

$= (2x^2 - 4)(2x)$

$\frac{dy}{dx} = 4x^3 - 8x$

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Example 2: Given $y = f(u) = u^3 - 5u$ where $u = g(x) = x^4 + 3x$

Solution:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$(3u^2 - 5)(4x^3 + 3)$$

Substitute $x^4 + 3x$ for u

$$\frac{dy}{dx} = [3(x^4 + 3x)^2 - 5](4x^3 + 3)$$

$$[3(x^8 + 6x^5 + 9x^2) - 5](4x^3 + 3)$$

$$(3x^8 + 18x^5 + 27x^2 - 5)(4x^3 + 3)$$

$$12x^{11} + 9x^8 + 72x^8 + 54x^5 + 108x^5 + 81x^2 - 20x^3 - 15$$

$$12x^{11} + 81x^8 + 162x^5 - 20x^3 + 81x^2 - 15$$

Practice Exercise

Find the derivative of each of the following

1a. y = 63

b. $y = x^{14}$

c. $y = 3x^4$

2a. $y = x^3 - 4x$

b. $y = \frac{x^6}{3} - 2x$

c. $y = 4x^0$

3a. $y = (x^3 - 2x)(x^5 + 6x^2)$

b. $y = (2 - x - 3x^4)(10 + x - 4x^3)$

c. $y = \left(\frac{x^2}{2} - 10\right)(x^3 - 2x^2 + 1)$

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4a. $y = (5x^3 + 1)^4$

b. $y = \sqrt{1 - 5x^3}$

c. $y = (x^2 - 2x + 5)^{1/3}$

In the following exercises, find $\frac{dy}{dx}$

5. $y = f(u) = u^3$ and $u = g(x) = x^2 + 3x + 1$

6. $y = f(u) = u^4 + u^2 + 1$ and $u = g(x) = x^2 - 4$

7. $y = f(u) = \sqrt{u}$ and $u = g(x) = \frac{x^2}{2}$

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LECTURE SIX

Some Economic Applications of the Rules

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Marginal Cost and Marginal Revenue

In general, the derivative of any total function gives its marginal function. For example, marginal cost is the derivative of total cost. i.e.

$$MC(q) = \frac{dTC(q)}{dq}$$

And $MR(q) = \frac{dTR(q)}{dq}$

e.g Given a short run total cost (TC) function

$$TC = Q^3 - 4Q^2 + 10Q + 75$$

the marginal cost function is the derivative of the TC function

$$MC(Q) = \frac{dTC}{dQ} = 3Q^2 - 8Q + 10$$

We would observe that the constant, which represents the fixed cost, does not affect the derivative because the derivative of a constant is zero. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost.

Finding Marginal Revenue Function from Average Revenue Function

Consider an average revenue (AR) function in the form AR = 15 – Q, we can find the marginal revenue function only after we have obtained the total revenue function.

$$TR(Q) = AR(Q)$$
$$TR(Q) = (15 - Q)Q$$
$$TR(Q) = 15Q - Q^2$$

We can then take the derivative of TR (Q) with respect to Q to get the MR function.

$$MR(Q) = \frac{dTR(Q)}{dQ} = 15 - 2Q$$

The same procedure applies to deriving the marginal cost function from the average cost function. First, obtain the Total-Cost (TC) by multiplying the Average Cost (AC) by Q i.e TC = AC × Q, then take the derivative of the TC to get MC.

$$MC = \frac{dTC}{dQ}$$

Observe the similarities and differences between the Average Revenue function and the Marginal Revenue function derived from it.

The two functions have the same vertical intercept (15) but the slope of the MR function is twice that of the AR function. This is a general result whenever we have a linear Average Revenue function (i.e. Total Revenue is a quadratic function)

GRAPH 7: Relationship between AR and MR in an imperfect market arrangement

For a firm in a perfectly competitive market arrangement, the Total Revenue function is linear; hence, the Marginal Revenue and the Average Revenue functions coincide at a constant.

Example: Consider a perfectly competitive firm with a Total Revenue function of the form TR(Q) = 32Q, find the AR and MR.

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Solution:

$$AR = \frac{TR}{Q}$$

$$AR = \frac{32Q}{Q} = 32$$

$$MR = \frac{dTR}{dQ} = 32$$

Therefore, for a firm in the perfect market arrangement, MR = AR = P

GRAPH 8

Relationship between Marginal cost and Average cost functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a Total Cost function TC = C(Q), the average cost (AC) function will be a quotient of two functions of Q, since

$AC = \frac{TC}{Q} = \frac{C(Q)}{Q}$ as long as Q > 0

Therefore, the rate of change of AC with respect to Q can be found by differentiating AC.

$$\frac{dAC}{dQ} = \frac{d\,{}^{C(Q)}/_{Q}}{dQ} = \frac{[C'(Q).Q - C(Q).1]}{Q^2}$$

We can split the fraction and have

$$\frac{[C'(Q).Q]}{Q^2} - \frac{[C(Q).1]}{Q^2}$$

$$\left[\frac{C'(Q)}{Q} - \frac{C(Q)}{Q^2}\right]$$

Factorize;

$$\frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

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Where $C'(Q) = MC$

And $\dfrac{C(Q)}{Q} = AC$

It then follows that

$$\frac{dAC}{dQ}\left|\begin{matrix}>\\=\\<\end{matrix}\right|0 \text{ if } C'(Q)\left|\begin{matrix}>\\=\\<\end{matrix}\right|\frac{C(Q)}{Q}$$

What this means is that;

The slope of AC is positive (i.e AC curve is upward sloping) when MC > AC

The slope of AC is zero when MC = AC; and

The slope of AC is negative when MC < AC

GRAPH 9

Observe that up to point Q* AC is downward sloping and the MC lies below it. At point Q*, the slope of AC = zero and MC = AC. At all points after Q*, AC is upward sloping and MC lies above it.

Example 1: Given the total cost function $TC = Q^3 - 8Q^2 + 15Q + 30$, write out a variable cost (VC) function. Find the derivative of the VC function, and give the economic meaning of that derivative.

Solution:

TC = VC + FC

$VC = Q^3 - 8Q^2 + 15Q$

$$\frac{dVC}{dQ} = 3Q^2 - 16Q + 15$$

The economic meaning of the derivative above is the marginal cost (MC)

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Remember we said that the fixed cost (FC) does not affect the MC. Therefore, the derivatives of both TC and VC give MC.

Example 2:

Given the average cost function $AC = 2Q^2 - 4Q + 214$, find the MC function. Is the given function more appropriate as a long-run or short-run function? Why?

Solution:

TC = AC × Q

$$TC = (2Q^2 - 4Q + 214)Q$$

$$TC = 2Q^3 - 4Q^2 + 214Q$$

$$MC = \frac{dTC}{dQ} = 6Q^2 - 8Q + 214$$

The function is more appropriate as a long-run function because the total cost (TC) function does not have any fixed component which is characteristic of the long-run.

Note: In the long-run, all factors of production are variable as well as the associated cost. Therefore, in the long-run, fixed cost = 0 and variable cost is equal to the total cost.

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Example 3:

Given AR = 60 – 5Q,

a. find the total revenue function and the marginal revenue function.
b. Comparing the AR and the MR functions, what can you say about their relative slopes?

Solution:

a.

$$TR = AR \times Q$$
$$TR = (60 - 5Q)Q$$
$$TR = 60Q - 5Q^2$$
$$MR = \frac{dTR}{dQ} = 60 - 10Q$$

b. The slope of the MR function (10) is twice the slope of the AR function.

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LECTURE SEVEN

Optimization: Maxima and Minima

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An introduction to higher order derivatives:

So far we have dealt with first-order derivatives, now we turn to higher-order derivatives.

Given a function f(x), there are other derivatives which can be defined. This section discusses these higher order derivatives.

The second derivative

The derivative $f'(x)$ of the function f(x) is often referred to as the first derivative of the function. The adjective first is used to distinguish this derivative from other derivatives associated with the function.

The second derivative $f''(x)$ or $\frac{d^2y}{dx^2}$ of a function is the derivative of the first derivative. This is done by applying the same rules of differentiation as were used in finding the first derivative.

Here are some examples. Table 1

	Function f(x)	**First derivative** $f'(x)$	**Second derivative** $f''(x)$
1	x^5	$5x^4$	$20x^3$
2	x^3-2x^2+5x	$3x^2-4x+5$	$6x-4$
3	$x^{3/2}$	$\frac{3}{2}x^{1/2}$	$\frac{3}{4}x^{-1/2}$
4	$(2x-10)^3$	$6(2x-10)^2$	$48x-240$

Let us solve the 4th example in Table 1 for clarity.

$y = (2x - 10)^3$ remember this is a power function.

$f(x) = [u(x)]^n$

$f'(x) = n[u(x)]^{n-1}.u'(x)$

$f'(x) = 3(2x - 10)^{3-1}.2$

$f'(x) = 6(2x - 10)^2$

To find $f''(x)$, differentiate the result in $f'(x)$

$f''(x) = 6(2)[2x - 10].2$

$f''(x) = 12(2x - 10).2$

$f''(x) = 24(2x - 10)$

$f''(x) = 48x - 240$

Exercise: Find the first four derivatives of the function $\frac{x}{1+x}$ where $x \neq -1$

Optimization

This involves maximizing or minimizing a function depending on the economic purpose it serves.

i. ***First-order condition for optimization (Necessary condition)***

For any function to reach its optimum point, the first derivative must be equal to zero i.e.

$$f''(x) = \frac{d^2y}{dx^2} > 0 \quad f'(x) = \frac{dy}{dx} = 0[\max or \min]$$

However, with the first order condition, we do not know the nature of the optimum point (either maximum or minimum). The second order condition helps to do this.

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ii. ***Second order condition for optimization (Sufficient condition)***

The nature of the optimum point is determined by taking the second derivative, if the second derivative is negative, we have a maximum point.

i.e $f''(x) = \frac{d^2y}{dx^2} < 0$ (maximum)

However if the second derivative is positive, then we have a minimum point. Thus

$f''(x) = \frac{d^2y}{dx^2} > 0$ (minimum)

Example 1: Find the relative extremum of the function $y = f(x) = 4x^2 - x$ and determine its nature.

Solution:

$\frac{dy}{dx} = 0$ (first order condition/necessary condition)

$$\frac{dy}{dx} = 8x - 1 = 0$$

$$8x = 1$$

$$x = 1/8$$

For the nature of the point, we differentiate again

$\frac{d^2y}{dx^2} = 8$ since the second derivative (8) is positive, we conclude that it is a minimum point.

Example 2: Find the relative extrema of the function $y = g(x) = x^3 - 3x^2 + 2$ and determine the nature of those points.

Solution:

$$\frac{dy}{dx} = g'(x) = 0$$

$$\frac{dy}{dx} = 3x^2 - 6x = 0$$

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solving the quadratic equation, we have

$$3x(x-2)=0$$

$$3x=0 \text{ or } x-2=0$$

$$x=0 \text{ or } x=2$$

The extrema occur at x = 0 and x = 2.

For their nature, we take the second derivative

$$\frac{d^2y}{dx^2}=6x-6$$

when $x=0$

$$=6(0)-6$$

$$\frac{d^2y}{dx^2}=-6$$

\- 6 < 0 (maximum)

when x = 2,

$$=6(2)-6$$

$$=12-6$$

$$\frac{d^2y}{dx^2}=6>0$$

Application to Revenue, Cost and Profit

Revenue maximization:

The money which flows into an organization either from selling products or providing services is referred to as revenue. The most fundamental way of computing total revenue from selling a product (or service) is

TR = Price per unit × Quantity sold.

Example:

The demand for the product of a firm varies with the price that the firm charges for the product. The firm estimates that annual total revenue is a function of price.

$$R=f(p)=-50p^2+500p$$

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a. Determine the price which should be charged in order to maximize total revenue
b. What is the maximum value of annual total revenue?

Solution:

a.

$$\frac{dR}{dp} = f'(p) = -100p + 500$$

if we set $f(P) = 0$

$$-100p + 500 = 0$$

$$-100p = -500$$

$$p = 5$$

the sufficient condition should be tested, thus

$$\frac{d^2R}{dp^2} = f''(p) = -100 < 0$$

b. the maximum value of R is found by substituting P = 5 into the function [f(p)]

$$f(5) = -50(5^2) + 500(5)$$

$$= -50(25) + 2500$$

$$= -1250 + 2500$$

$$R_m = 1250$$

where R_m means maximum revenue.

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Cost minimization: Producers attempt to be efficient by minimizing cost.

Example 1: A retailer of motorized bicycles has examined cost data and has determined a cost function which expresses the annual cost of purchasing, owning, and maintaining inventory as a function of the size (number of units) of each order it places for the bicycles. The cost function is

$$C = f(q) = \frac{4860}{q} + 15q + 750000$$

a. Determine the order size which minimizes annual inventory cost
b. What is the minimum annual inventory cost expected to be?

Solution:

We may rewrite the function as

$$f(q) = 4860q^{-1} + 15q + 750000$$

$$\frac{dC}{dq} = f'(q) = -4860q^{-2} + 15$$

if we equate $f'(q) = 0$, we have

$$-4860q^{-2} + 15 = 0$$

$$-4860q^{-2} = -15$$

$$\frac{-4860}{q^2} = -15$$

multiply both sided by q^2

$$-4860 = -15q^2$$

divide both sides by -15

$$324 = q^2$$

take the square root of both sides

$$\sqrt{324 = \sqrt{9^2}}$$

$$q = \pm 18$$

the value q = - 18 is meaningless in this application (it is not possible to have a negative quantity of goods). We can check the nature of the critical (extremum) point at q = 18.

$$\frac{d^2C}{dq^2} = f''(q) = 9720q^{-3}$$

$$= \frac{9720}{q^3}$$

substitute q = 18

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$$\frac{9720}{18^3} = 1.667 > 0(\min)$$

b. to get the minimum inventory cost, we substitute 18 for q in the original cost function.

$$f(18) = \frac{4860}{18} + 15(18) + 750000$$
$$= 270 + 270 + 750000$$
$$= 750540$$

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Example 2: The total cost of producing q units of a certain product is described by the function $C = 100000 + 1500q + 0.2q^2$

a. Determine the number of units of q that should be produced in order to minimize average cost per unit.
b. Show that MC = AC at the minimum point of AC.

Solution:

a. we are asked to minimize average cost, not total cost. So we need to find the AC function from the TC function given.

$$AC = \frac{TC}{Q}$$
$$AC = \frac{100000}{q} + 1500 + 0.2q$$
$$AC = 100000q^{-1} + 1500 + 0.2q$$
$$\frac{dAC}{dq} = -100000q^{-2} + 0.2$$
$$\frac{dAC}{dq} = 0$$
$$\frac{-100000}{q^2} + 0.2 = 0$$
$$\frac{-100000}{q^2} = -0.2$$

Multiply both sides by q^2

$$-100000 = -0.2q^2$$

Divide both sides by – 0.2

$$q^2 = \frac{-100000}{-0.2} = 500000$$

$$q = \sqrt{500000}$$ take the square root of both sides

$$q = \pm 707.11$$

Negative quantity is not attainable, so we take q = 707.11 as the number of units required to minimize AC.

The nature of this critical point can be tested, thus

$$\frac{d^2AC}{dq^2} = 200000q^{-3}$$

$$= \frac{200000}{q^2}$$

$$= \frac{200000}{(707.11)^3}$$

$$= 0.00056 > 0(\text{min})$$

b.

$$MC = \frac{dTC}{dq} = 1500 + 0.4q$$

$$AC = \frac{100000}{q} + 1500 + 0.2q$$

Substitute the value of q at which AC is minimum i.e. 707.11 into both MC and AC functions and solve

$$MC = 1500 + 0.4(707.11) = 1782.84$$

$$AC = \frac{100000}{707.11} + 1500 + 0.2(707.11) = 1782.84$$

It is shown that MC = AC = 1782.84 at the point where AC is minimum.

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GRAPH 10

Practice Exercise

1. A firm has determined that total revenue is a function of the price charged for its product. Specifically, the total revenue function is $R = f(p) = -10p^2 + 1750p$

   a. Determine the price which results in the maximum total revenue
   b. What is the maximum value of total revenue?

2. A community which is located in a resort area is trying to decide on the parking fee to charge at the town-owned beach. There are other beaches in the area, and there is competition for bathers among the different beaches. The town has determined the following function which expresses the average number of cars per day 'q' as a function of the parking fee 'p'

   Q = 600 – 12P

   a. Determine the fee which should be charged to maximize daily beach revenue.
   b. What is the maximum beach revenue expected to be?
   c. How many cars are expected on an average day?

The example below will be helpful in solving question 2.

The demand function for a firm's product is q = 150000 – 75p

a. Determine the fee which should be charged to maximize daily beach revenue.
b. What is the maximum beach revenue expected to be?
c. How many cars are expected on an average day?

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Solution:

a.

$$TR = P \times Q$$

$$TR = (150000 - 75p)p$$

$$TR = 150000 - 75p^2$$

To maximize, $\frac{dTR}{dp} = 0$

$$\frac{dTR}{dp} = 150000 - 150p = 0$$
$$150000 = 150p$$
$$p = \frac{150000}{150}$$
$$p = 1000$$

b. the maximum value of TR is obtained by substituting 1000 for p in the TR function.

TR = P × Q

$$TR = 150000(1000) - 75(1000)^2$$
$$TR = 150000000 - 75000000$$
$$TR = 75000000$$

c. Substitute 1000 for p in the demand function.

$$q = 150000 - 75(1000)$$
$$q = 150000 - 75000$$
$$q = 75000$$

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LECTURE EIGHT

Profit Maximization/Loss Minimization

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Let us employ the general form of total revenue R = R (Q) and total cost C = C(Q). we know that profit is equal to revenue minus cost

i.e.

$$\prod = R - C$$

$$\prod(Q) = R(Q) - C(Q)$$

To maximize profit, we need to satisfy the necessary condition for a maximum

$$\frac{d\pi}{dQ} = 0$$

$$\frac{d\pi}{dQ} = R'(Q) - C'(Q)$$

$$R'(Q) - C'(Q) = 0$$

$$R'(Q) = C'(Q)$$

Where $R'(Q)$ = Marginal Revenue (derivative of Total Revenue),

and $C'(Q)$ = Marginal Cost (derivative of Total Cost)

therefore, for profit to be maximized,

MR = MC

However, the first order condition may lead to a minimum rather than a maximum; thus, we must check the second order condition;

$$\frac{d^2\pi}{dQ^2} = \prod''(Q) = R''(Q) - C''(Q)$$

For this to be less than zero (<0),

$$R''(Q) < C''(Q)$$

This implies that the slope of MR must be less than the slope of MC as the sufficient condition for profit maximization.

Example 1: A manufacturer has developed a new design for solar collection panels. Marketing studies have indicated that annual demand for the panels depend on the price charged. The demand function for the panels has been estimated as q = 100000-200p the total cost of producing 'q' panels is estimated by the function $C = 150000 + 100q + 0.003q^2$.

a. Write out the profit function for the manufacturer
b. Determine the units of output that should be produced to maximize profit
c. Determine the price that should be charged per unit
d. What is the maximum annual profit?

Solution:

a. $\prod(q) = TR(q) - TC(q)$

$$TR = p \times q$$

Because we want our total revenue to be expressed in terms of q, we make p the subject of the formula in the demand function.

$$200p = 100000 - q$$

Divide both sides by 200

$$p = 500 - 0.005q$$

$$TR = p \times q = (500 - 0.005q)q$$

$$TR = 500q - 0.005q^2$$

$$\prod = 500q - 0.005q^2 - (150000 + 100q + 0.003q^2)$$

$$\prod = 500q - 0.005q^2 - 150000 - 100q - 0.003q^2$$

$$\prod = -0.008q^2 + 400q - 150000$$

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b. $\frac{d\pi}{dq} = 0$

$$\frac{d\pi}{dq} = -0.016q + 400 = 0$$

$$0.016q = 400$$

$$q = \frac{400}{0.016}$$

$$q = 25,000$$

c. substitute 25,000 for q in the p function.

$$p = 500 - 0.005q$$

$$p = 500 - 0.005(25000)$$

$$p = 500 - 125$$

$$p = 375$$

e. The maximum profit (substitute for q in the profit function)

$$\prod = -0.008q^2 + 400q - 150000$$

$$= -0.008(25000)^2 + 400(25000) - 150000$$

$$\prod = 4,850,000$$

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Exercise:

1. A firm has the following total cost and demand functions

$$\frac{1}{3}Q^3 - 7Q^2 + 111Q + 50$$

$$Q = 100 - p$$

a. Write out the total revenue function in terms of Q
b. Formulate the total profit function in terms of Q
c. Find the profit maximizing level of output
d. What is the maximum profit?

2. A company estimates that demand for its product fluctuates with the price it charges. The demand function is q = 280000 – 400p. The total

cost of producing q units of the product is estimated by the function $C = 350000 + 300q + 0.0015q^2$

a. Determine how many units of q should be produced in order to maximize profit
b. What price should be charged?
c. What is the maximum profit expected?

Effects of Taxes and Subsidies on the Profit maximizing behaviour

There are basically two forms of taxes that can be imposed on a firm.

1. Lump-sum tax
2. Quantity/per unit tax

Taxes generally reduce the total revenue accruing to the firm.

Effects of Lump-sum tax

Suppose a lump-sum tax of 100,000 is imposed on the manufacturer in the example above:

a. how will it affect his profit maximizing behaviour (i.e. quantity and price)
b. what will his maximum profit be?

Solution:

Before tax, $TR = 500q - 0.005q^2$

After tax, $TR = 500q - 0.005q^2 - 100,000$

The total cost function remains unchanged

$$\prod = TR - TC$$

$$\prod = 500q - 0.005q^2 - 100000 - (150000 + 100q + 0.003q^2)$$

$$\prod = 500q - 0.005q^2 - 100000 - 150000 - 100q - 0.003q^2$$

$$\prod = 400q - 0.008q^2 - 250000$$

$$\frac{d\pi}{dq} = 0; 400 - 0.016q = 0$$

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$$0.016q = 400$$
$$q = 25,000$$
$$p = 500 - 0.005q$$
$$p = 500 - 0.005(25000)$$
$$p = 500 - 125$$
$$p = 375$$

The lump-sum tax does not affect the profit maximizing level of output and the price charged.

b. $\prod = 400q - 0.008q^2 - 250000$

$$= 400(25000) - 0.008(25000)^2 - 250000$$
$$= 10,000,000 - 5,000,000 - 250,000$$
$$= 4,750,000$$

Effects of Quantity/Unit tax

Suppose a per unit tax of 10 is imposed on the producer above,

a. Examine the effect on profit maximizing level of output
b. What price will be charged?
c. What is the new profit?

Solution:

Before tax, $TR = 500q - 0.005q^2$

After tax, $TR = 500q - 0.005q^2 - 10q$

$$TR = 490q - 0.005q^2$$

Total cost is not affected

$$\prod = 490q - 0.005q^2 - (150000 + 100q + 0.003q^2)$$
$$= 490q - 0.005q^2 - 150000 - 100q - 0.003q^2$$
$$\prod = 390q - 0.008q^2 - 150000$$

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$$\frac{d\pi}{dq} = 0; 390 - 0.016q = 0$$

$$0.016q = 390$$

$$q = 24,375$$

$$p = 500 - 0.005q$$

$$p = 500 - 0.005(24,375)$$

$$p = 500 - 121.875$$

$$p = 378.125$$

Substituting q = 24,375 in the equation

$$\prod = 390q - 0.008q^2 - 150000$$

$$= 390(24375) - 0.008(24375)^2 - 150000$$

$$= 4,603,125$$

The quantity tax reduces the profit maximizing level of output, increases the price and reduces the maximum profit.

"The effect of subsidies is the direct opposite of the observation above. This is because subsidy is a negative tax."

Differentiation and point elasticity

Consider a demand function Q = f(P), Elasticity is defined as

$$\frac{\Delta Q}{Q} \div \frac{\Delta p}{p} = \frac{\%\Delta Q}{\%\Delta p}$$

For infinitesimal change,

$$\Delta Q = dQ; \Delta p = dp$$

$$\in = \frac{dQ}{Q} \div \frac{dp}{p}$$

$$\in = \frac{dQ}{dp} \times \frac{p}{Q}$$

Example: Find the price elasticity of demand for Q = 100 – 2P when P = 10.

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Solution:

$$\in = \frac{dQ}{dp} \times \frac{p}{Q}$$

$$\frac{dQ}{dp} = -2;\ p = 10$$

$$Q = 100 - 2(10) = 80$$

$$\in = -2 \times \frac{10}{80}$$

$$\in = -0.25$$

For the interpretation, we only take the absolute value $|\in| = 0.25 < 1 (inelastic)$

$$\begin{bmatrix} Elastic \\ UnitElastic \\ Inelastic \end{bmatrix} \text{when } \in \begin{bmatrix} > \\ = \\ < \end{bmatrix} 1$$

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LECTURE NINE

Multivariate Differential Calculus

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Introduction

When a function has more than one variable, there are two types of differentiation that can be considered – partial and total. The total derivative allows all variables to change simultaneously while partial derivative allows only one variable to change, holding others constant.

Partial derivative assumes that the variables of the function are independent of one another and so other variables are treated as constant when considering a particular variable. $y = f(u,v)$; the partial derivatives are given by $\frac{dy}{du} = \lim_{\Delta u \to 0}\left(\frac{\Delta y}{\Delta u}\right)$. E.g.

$$y = 3u^2 - 3v$$

$$\frac{\partial y}{\partial u} = 6u; \frac{\partial y}{\partial v} = -3$$

Observe the difference in notation compared with what we had in single variable functions.

Single variable function	**Multi-variate function**
$y = f(u)$	$y = f(u,v,...)$
$\frac{dy}{du}$ or $f'(u)$	$\frac{\partial y}{\partial u}$ or f_u; $\frac{\partial y}{\partial v}$ or f_v
$\frac{d^2y}{du^2}$ or $f''(u)$	$\frac{\partial^2 y}{\partial u^2}$ or f_{uu}; $\frac{\partial^2 y}{\partial v^2}$ or f_{vv}

Example 1

Find all the partial derivatives of the function $y = (u + 2v + w^2)^2$.

Solution:

To find $\frac{\partial y}{\partial u} = f_u$ treat $2v + w^2$ as a constant

$$y = (u + c)^2$$

$$\frac{\partial y}{\partial u} = 2(u + c)$$

$$\frac{\partial y}{\partial u} = 2(u + v + w^2)$$

For $\frac{\partial y}{\partial v}$, treat $u + w^2$ as a constant

$$y = (2v + c)^2$$

$$\frac{\partial y}{\partial v} = 2(2v + c).2$$

$$\frac{\partial y}{\partial v} = 4(2v + u + w^2)$$

Lastly, for $\frac{\partial y}{\partial w}$, treat u + 2v as a constant

$$y = (c + w^2)$$

$$\frac{\partial y}{\partial w} = 2(c + w^2).2w$$

$$= 4w(u + 2v + w^2)$$

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Example 2

Find f_x and f_y if $f(x, y) = -10xy^3$

Solution: To find f_x, y must be assumed to be constant. This function can then be mentally rearranged as $f(x, y) = (-10y^3)x$

Where $-10y^3$ is the constant coefficient of x.

Therefore, $f_x = -10y^3$

For f_y, the function can be assumed as having the form $f(x, y) = (-10x)y^3$

Where $-10x$ is the constant coefficient of y^3

Therefore, $f_y = 3(-10x)y^2$

$f_y = -30xy^2$

Second-order partial derivatives

1. Pure second-order partial derivative; and
2. Cross/mixed partial derivative

Example 1: Given that a firm's costs are related to output of two goods x and y in the form $TC = x^2 - 0.5xy + 2y^2$ ……………………………(i)

It is possible to find the first-order partial derivatives f_x and f_y which translate to the additional cost of a slight increment in x and y respectively.

$f_x = 2x - 0.5y$ ……………………………………….. (ii)

$f_y = -0.5y + 4y$ …………………………………… (iii)

It is possible to find three different second-order partial derivatives. i.e.

f_{xx}
} pure second-order partial derivatives
f_{yy}

$f_{xy} = f_{yx}$ Cross partial derivative

Note: The cross partial f_{xy} is always equal to f_{yx}. This proposition is known as the *Young's theorem*

To obtain f_{xx}, differentiate equation (ii) with respect to x

$f_{xx} = 2$

For f_{yy}, differentiate equation (iii) with respect to y

$f_{yy} = 4$

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Lastly, for f_{xy} differentiate equation (ii) with respect to y

$f_{xy} = -0.5$ the same result is obtained if we differentiate equation (iii) with respect to x

Optimization of Multi-variate functions in Economics

Many producers or sellers deal in more than one product item and in order to maximize profit, or minimize cost, they need an optimal mix of the products in which they deal. Optimization of multivariate functions helps to achieve this.

Conditions for optimality:

Condition	Maxima	Minima
1st order	$f_x = 0,\ f_y = 0$	$f_x = 0,\ f_y = 0$
2nd order	$f_{xx} < 0,\ f_{yy} < 0$	$f_{xx} > 0,\ f_{yy} > 0$
3rd order	$(f_{xx})(f_{yy}) > (f_{xy})^2$	$(f_{xx})(f_{yy}) > (f_{xy})^2$

Example 1: A firm producing two goods x and y has the profit function $\prod = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$. To find the profit maximizing level of output for each of the two goods, and test to be sure that profits are maximized,

1. Take the first-order partial derivatives, set them equal to zero and solve for x and y simultaneously.

$$\prod_x = 64 - 4x + 4y = 0$$

$$\prod_y = 4x - 8y + 32 = 0$$

$$-4x + 4y = -64 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$4x - 8y = -32 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Add equations (i) and (ii)

$$-4y = -96$$

$$y = \frac{-96}{-4} = 24$$

To get x,

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$$4x - 8(24) = -32$$
$$4x - 192 = -32$$
$$4x = -32 + 192$$
$$4x = 160$$
$$x = 40$$

2. Take the second order direct partial derivatives and make sure both are negative, as required for a relative maximum.

$$\prod_{xx} = -4 < 0$$
$$\prod_{yy} = -8 < 0$$

3. Take the cross partials to make sure that $\prod_{xx} \prod_{yy} > (\prod_{xy})^2$

$$(-4)(-8) > (4)^2$$
$$32 > 16$$

We have now confirmed that profit is indeed maximized.

We can also find the profit at that point.

$$\prod = 64x - 2x^2 + 4xy - 4y^2$$
$$= 64(40) - 2(40)^2 + 4(40)(24) - 4(24)^2 + 32(24) - 14$$
$$= 2560 - 3200 + 3840 - 2304 + 768 - 14$$
$$\prod = 1650$$

Example 2: In monopolistic competition, producers must determine the price that will maximize their profit. Assume that a producer offers two different brands of a product, for which the demand functions are

$$Q_1 = 14 - 0.25P_1$$
$$Q_2 = 24 - 0.5P_2$$

And the joint cost function is $TC = Q_1^2 + 5Q_1Q_2 + Q_2^2$

Find the profit maximizing level of output, the price that should be charged for each brand, and the maximum profit.

Solution: First establish the profit function in terms of Q_1 and Q_2

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Since $\prod = TR - TC$, we need to find the firm's TR

$$TR = P_1Q_1 + P_2Q_2$$

Make P_1 and P_2 the subject in the demand functions.

$$P_1 = 56 - 4Q_1$$

$$P_2 = 48 - 2Q_2$$

$$TR = (56 - 4Q_1)Q_1 + (48 - 2Q_2)Q_2$$

$$TR = 56Q_1 - 4Q_1^2 + 48Q_2 - 2Q_2^2$$

$$\prod = 56Q_1 - 4Q_1^2 + 48Q_2 - 2Q_2^2 - Q_1^2 - 5Q_1Q_2 - Q_2^2$$

$$\prod = 56Q_1 - 5Q_1^2 + 48Q_2 - 3Q_2^2 - 5Q_1Q_2$$

We can now maximize the profit function

$$\frac{\partial \pi}{\partial Q_1} = 56 - 10Q_1 - 5Q_2$$

$$\frac{\partial \pi}{\partial Q_2} = 48 - 6Q_2 - 5Q_1$$

$10Q_1 + 5Q_2 - 56 = 0$ ……………………(i)

$5Q_1 + 6Q_2 - 48 = 0$ …………………….(ii)

Multiply equation ii by 2 to eliminate Q_1

$10Q_1 + 12Q_2 - 96 = 0$ ………………….(iii)

Subtract equation (i) from (iii)

$$7Q_2 - 40 = 0$$

$$7Q_2 = 40$$

$$Q_2 = 5.7$$

Substitute $Q_2 = 5.7$ into equation (i) or (ii)

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$$10Q_1 + 5Q_2 - 56 = 0$$
$$10Q_1 + 5(5.7) - 56 = 0$$
$$10Q_1 + 28.5 - 56 = 0$$
$$10Q_1 = 27.5$$
$$Q_1 = 2.75$$

Take the second derivatives to be sure profit is maximized

$$\prod_{Q_1Q_1} = -10 < 0 \; ; \; \prod_{Q_2Q_2} = -10 < 0 \; ; \; \prod_{Q_1Q_2} = -5$$

Therefore, $\prod_{Q_1Q_1} \prod_{Q_2Q_2} < (\prod_{Q_1Q_2})^2$

To find the profit maximizing prices

$$P_1 = 56 - 4Q_1$$
$$= 56 - 4(5.7)$$
$$P_1 = 45$$
$$P_2 = 48 - 2Q_2$$
$$= 48 - 2(5.7)$$
$$P_2 = 36.6$$

Lastly, the maximum profit

$$\prod = 56Q_1 - 5Q_1{}^2 + 48Q_2 - 3Q_2{}^2 - 5Q_1Q_2$$
$$= 56(2.75) - 5(2.75)^2 + 48(5.7) - 3(5.7)^2 - 5(2.75)(5.7)$$
$$\prod = 213.94$$

Exercise:

1. Given the profit function $\prod = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18$ for a firm producing two goods x and y,
   a. Find the levels of output x and y at which profit is maximized
   b. Test the second order conditions
   c. What is the maximum profit?
2. A monopolist sells two products x and y for which the demand functions are

   $x = 25 - 0.5P_x$

   $y = 30 - P_y$

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And the combined cost function is $C = x^2 + 2xy + y^2 + 20$. Find,

a. The profit maximizing levels of output
b. The profit maximizing prices
c. The maximum profit.

3. Find:
   a. profit maximizing levels of output
   b. Prices and
   c. maximum profit when $Q_1 = 520 - P_1$

$$Q_2 = 820 - 2P_2$$

And $C = 0.1Q^2 + 0.1Q_1Q_2 + 0.2Q_2^2 + 325$

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LECTURE TEN

Constrained Optimization: The Lagrange Multiplier Method

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Solutions to economic problems usually have to be found under constraints (e.g. maximizing utility subject to budget constraint or minimizing cost subject to minimal requirement of output). The use of Lagrangian function helps to solve this.

The constraints represent restrictions that can influence the degree to which an objective function is optimized. Constraints may reflect such restrictions as limited resources (e.g. labour, materials or capital), limited demand for products, sales goals etc. problems having this structure are considered to be constrained optimization problems.

Generally, we can optimize $y = f(x_1x_2)$ subject to $g(x_1x_2) = K$

Where $f(x_1x_2)$ is the objective function, and

$g(x_1x_2)$ is the constraint.

To solve, we set up a composite function known as the Lagrangian function. Thus

$$L(x_1x_2\lambda) = f(x_1x_2) - \lambda[g(x_1x_2) - K]$$

The variable λ (lambda) is known as the lagrangian multiplier.

Notice in the lagrangian function that λ can equal any value and the term $\lambda[g(x_1x_2) - K]$ will equal zero, provided that (x_1x_2) are values which satisfy the constraint.

Then we take the partial derivatives of $L(x_1x_2\lambda)$ with respect to x_1, x_2 and λ and set them equal to 0. i.e.

$L_{x_1} = 0$; $L_{x_2} = 0$; and $L_\lambda = 0$

We then solve the equations simultaneously.

Example1: Maximize $f(x_1x_2) = 25 - x_1^2 - x_2^2$ subject to $2x_1 + x_2 = 4$

Solution: The lagrangian function yields the following transformation

$$L(x_1x_2\lambda) = 25 - x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 4)$$

Taking the partial derivatives, we have

$$L_{x_1} = -2x_1 - 2\lambda$$

$$L_{x_2} = -2x_2 - \lambda$$

$$L_\lambda = -2x_1 - x_2 + 4$$

Critical values are found by setting the three partial derivatives equal to zero.

$-2x_1 - 2\lambda = 0$ …………………..(i)

$-2x_2 - \lambda = 0$ ……………………(ii)

$-2x_1 - x_2 + 4 = 0$ ………………..(iii)

Multiply equation ii by 2

$-4x_2 - 2\lambda = 0$ ……………………(iv)

Subtract equation (iv) from (i).

$$-2x_1 + 4x_2 = 0$$

$$2x_1 = 4x_2$$

$$x_1 = 2x_2$$

Substitute $x_1 = 2x_2$ into equation (iii)

$$-2(2x_2) - x_2 + 4 = 0$$

$$-4x_2 - x_2 + 4 = 0$$

$$-5x_2 = -4$$

$$x_2 = 0.8$$

Since $x_1 = 2x_2$,

$$x_1 = 2(0.8)$$

$$x_1 = 1.6$$

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To find λ, substitute into either equation (i) or (ii)

$$-2x_2 - \lambda = 0$$
$$\lambda = -2x_2$$
$$\lambda = -2(0.8)$$
$$\lambda = -1.6$$

The maximum value of the objective function can also be found.

$$f(x_1x_2) = 25 - x_1^{\,2} - x_2^{\,2}$$
$$= 25 - 1.6^2 - 0.8^2$$
$$= 25 - 2.56 - 0.64$$
$$= 21.8$$

What does λ (the lagrangian multiplier) mean?

λ shows the extent (and direction) to which the objective function changes when the constant part of the constraint changes by 1 unit. From the example above, if the constant in the constraint increases by 1 (changes from 4 to 5), the objective function will reduce by 1.6 (i.e. 21.8 – 1.6 = 20.2)

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Example 2:

a. What combination of goods x and y should a firm produce to minimize cost when the joint cost function is $C = 6x^2 + 10y^2 - xy + 30$ and the firm has a production quota of $x + y = 34$.

b. Estimate the effect on cost if the production quota is reduced by 1 unit.

Solution:

$$L = 6x^2 + 10y^2 - xy + 30 - \lambda(x + y - 34)$$
$$L_x = 12x - y - \lambda$$
$$L_y = 20y - x - \lambda$$
$$L_\lambda = -x - y + 34$$

Equate to zero and solve simultaneously

$12x - y - \lambda = 0$ …………………… (i)

$20y - x - \lambda = 0$ ……………………. (ii)

$-x - y + 34 = 0$ ……………………. (iii)

Subtract equation (i) from (ii).

$$-13x + 21y = 0$$
$$13x = 21y$$
$$x = \frac{21}{13}y$$

Substitute $x = \frac{21}{13}y$ into equation (iii)

$$-x - y + 34 = 0$$
$$-\frac{21}{13}y - y = -34$$
$$\frac{-21y - 13y}{13} = -34$$
$$-34y = -442$$
$$y = 13$$
$$x = \frac{21}{13}y; x = \frac{21}{13}(13)$$
$$x = 21$$

Solve for λ in equation (i)

$$12x - y - \lambda = 0$$
$$12(21) - 13 - \lambda = 0$$
$$252 - 13 = \lambda$$
$$\lambda = 239$$

The minimum value of cost

$$C = 6x^2 + 10y^2 - xy + 30$$
$$C = 6(21)^2 + 10(13)^2 - 21(13) + 30$$
$$C = 4093$$

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b. with $\lambda = 239$, a unit decrease in the constant of the constraint (the production quota) will lead to a cost reduction of approximately 239.

Example 3: A monopolistic firm has the following demand functions for each of its products x and y;

$$x = 72 - 0.5P_x$$
$$y = 120 - P_y$$

The combined cost function is $C = x^2 + xy + y^2 + 35$ and maximum joint production is 40, thus x + y = 40. Find the profit maximizing level of:

a. output
b. prices and
c. the maximum profit.

Solution:

$$P_x = 144 - 2x$$
$$P_y = 120 - y$$

$$TR_x = (144 - 2x)x \;\; ; \; TR_y = (120 - y)y$$

$$TR = TR_x + TR_y = (144 - 2x)x + (120 - y)y$$

$$\prod = 144x - 2x^2 + 120y - y^2 - (x^2 + xy + y^2 + 35)$$

$$\prod = 144x - 2x^2 + 120y - y^2 - x^2 - xy - y^2 - 35$$

$$\prod = 144x - 3x^2 - xy - 2y^2 + 120y - 35$$

With the constraint,

$$L = 144x - 3x^2 - xy - 2y^2 + 120y - 35 - \lambda(x + y - 40)$$

$$L_x = 144 - 6x - \lambda$$
$$L_y = -x - 4y + 120 - \lambda$$
$$L_\lambda = -x - y + 40$$

Equate to zero and solve simultaneously

$144 - 6x - \lambda = 0$ ……………………….(i)

$-x - 4y + 120 - \lambda = 0$ ………………..(ii)

$-x - y + 40 = 0$ ………………………(iii)

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Subtract equation i from ii

$$5x-3y-24=0 \text{ ………………………….(iv)}$$

Multiply equation (iii) by 3

$$-3x-3y+120=0$$

Subtract equation (iv) from (v)

$$-8x+144=0$$

$$8x=144$$

$$x=18$$

Substitute x = 18 into equation (iii)

$$-x-y+40=0$$

$$-18-y+40=0$$

$$-y+22=0$$

$$y=22$$

Substitute x = 18 and y = 22 into equation (i)

$$144-6(18)-22-\lambda=0$$

$$144-108-22-\lambda=0$$

$$14-\lambda=0$$

$$\lambda=14$$

b. $P_x=144-2x$

$$=144-2(18)$$

$$=108$$

$$P_y=120-y$$

$$=120-22$$

$$=98$$

c. $\prod=144x-3x^2-xy-2y^2+120y-35$

$$\prod=144(18)-3(18)^2-18(22)-2(22)^2+120(22)-35$$

$$\prod=2861$$

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Exercise:

1a. Minimize costs for a firm with the cost function $C = 5x^2 + 2xy + 3y^2 + 800$ subject to the production quota $x + y = 39$.

b. Estimate additional costs if the production quota is increased to 40

2a. Maximize utility $U = Q_1Q_2 + Q_1 + 2Q_2$ subject to $2Q_1 + 5Q_2 = 51$.

b. What happens to the consumer's utility if his budget changes from 51 to 56

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LECTURE ELEVEN

Constrained Optimization of Cobb-Douglas Functions

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Economic analysis frequently employs the Cobb-Douglas production function $q = AK^{\alpha}L^{\beta}(A > 0; 0 < \alpha, \beta < 1)$, where q is the quantity of output in physical units, K is quantity of capital and L is the quantity of labour. Here α measures the percentage change in q for a 1 percent change in K while L is held constant. β does the exact opposite. A is an efficiency parameter reflecting the level of technology.

A strict Cobb-Douglas function, in which $\alpha + \beta = 1$, exhibits constant returns to scale. For a Cobb-Douglas function, in which $\alpha + \beta \neq 1$, there is increasing returns to scale if $\alpha + \beta > 1$; and decreasing returns to scale if $\alpha + \beta < 1$.

Example 1: Given a budget constraint of ₦108 when $P_k = 3$ and $P_L = 4$, the generalized Cobb-Douglas production function $q = K^{0.4}L^{0.5}$ is optimized as follows

Solution:

The constraint $3K + 4L = 108$

Set up the Lagrangian function

$q = K^{0.4}L^{0.5} - \lambda(3K + 4L - 108)$

Take the first-order partial derivatives

$$\frac{\partial q}{\partial K} = 0.4K^{-0.6}L^{0.5} - 3\lambda$$

$$\frac{\partial q}{\partial L} = 0.5K^{0.4}L^{-0.5} - 4\lambda$$

$$\frac{\partial q}{\partial \lambda} = -3K - 4L + 108$$

Equate the derivatives to zero and solve simultaneously

$0.4K^{-0.6}L^{0.5} = 0$ ……………………………i

$0.5K^{0.4}L^{-0.5} = 0$ ……………………………ii

$3K + 4L = 108$ …………………………..iii

Divide equation (i) by (ii) to eliminate λ

$$\frac{0.4K^{-0.6}L^{0.5}}{0.5K^{0.4}L^{-0.5}} \quad \frac{3\lambda}{4\lambda}$$

$$0.8K^{-0.6-0.4}L^{0.5-(-0.5)} = \frac{3}{4}$$

$$0.8K^{-1}L^{1} = 0.75$$

$$K^{-1}L^{1} = \frac{0.75}{0.8}$$

$$\frac{L}{K} = 0.9375$$

$$L = 0.9375K$$

Substitute $L = 0.9375K$ into equation iii

3K + 4(0.9375K) = 108

3K + 3.75K = 108

6.75K = 108

K = 16

Substitute K = 16 into L = 0.9375K

$L = 0.9375(16)$

$L = 15$

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Example 2: Optimize $q = K^{0.3}L^{0.5}$ subject to $6K + 2L = 384$

Solution:

$$q = K^{0.3}L^{0.5} - \lambda(6K + 2L - 384)$$

$$\frac{\partial q}{\partial K} = 0.3K^{-0.7}L^{0.5} - 6\lambda = 0 \text{ ……………………….(i)}$$

$$\frac{\partial q}{\partial L} = 0.5K^{0.3}L^{-0.5} - 2\lambda = 0 \text{ ……………………….(ii)}$$

$$\frac{\partial q}{\partial \lambda} = -6K - 2L + 384 = 0$$

$$6K + 2L = 384 \text{ …………………………(iii)}$$

Divide equation (i) by (ii)

$$\frac{0.3K^{-0.7}L^{0..5}}{0.5K^{0.3}L^{-0.5}} = \frac{6\lambda}{2\lambda}$$

$$0.6K^{-1}L^{1} = 3$$

$$K^{-1}L^{1} = \frac{3}{0.6}$$

$$\frac{L}{K} = 5$$

$$L = 5K$$

Substitute L = 5K into equation (iii)

$$6K + 2(5K) = 384$$

$$6K + 10K = 384$$

$$16K = 384$$

$$K = 24$$

$$L = 5K$$

$$L = 5(24)$$

$$L = 120$$

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Exercise: Maximize the following utility functions subject to the given budget constraints.

a. $U = X^{0.6}Y^{0.25}$ given that $P_x = 8$; $P_y = 5$ and $B = 680$

b. $U = X^{0.8}Y^{0.2}$ given that $P_x = 5$; $P_y = 3$ and $B = 75$

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LECTURE TWELVE

Integral Calculus

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Introduction

Integration, also known as anti-derivative, is the reverse of differentiation. The function to be integrated is known as the integrand while the result is referred to as the integral. There are two major categories of integral calculus, these are; indefinite and definite integral.

Rules of Integral Calculus

1. ***Constant function Rule***: The integral of any constant K is given by $\int Kdx = Kx + C$ where C is a constant

   Example 1: $\int 2dx = 2x + C$

   Example 2: $\int 35dx = 35x + C$

   Example 3: $\int 1dx = x + C$

2. ***The Power Rule***: If y is a function of x, and x is a power function, as in $y = f(x) = x^n$, then

   $$\int f(x)dx = \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

   Example 1: $f(x) = x^4$

   $$\int f(x)dx = \int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$$

   Example 2: Find the integral of $f(x) = \frac{1}{x^4}$.

Solution: The function can be rewritten as $f(x) = x^{-4}$

$$\int f(x)dx = \int x^{-4}dx = \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C$$

3. **Logarithmic Rule:** $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$

Example 1: $\int \frac{1}{x} dx = \ln x + C$

Example 2: $\int \frac{3}{x} dx = 3\int \frac{1}{x} = \ln x + C$

Example 3: $\int \frac{3}{x^2+3} dx \ln(x^2+3)\ C$

Example 4: $\int \frac{6x}{x^2+12} dx = 3\int \frac{2x}{x^2+12} dx = 3\ln x + C$

4. **Composite function Rule:** If $y = [f(x)]^n\, g(x)$

where g(x) is related to $f'(x)$ before or after factorization,

$$\int y dx = [f(x)]^n\, g(x) = \frac{[f(x)]^{n+1}}{n+1} \cdot \frac{g(x)}{f'(x)} + C$$

Example 1: $\int (x^3 + 3x^2 + 1)^3 (x^2 + 2x) dx$

Since $f(x) = x^3 + 3x^2 + 1;\ g(x) = x^2 + 2x$

$f'(x) = 3x^2 + 6x = 3(x^2 + 2x)$

$$\int (x^3 + 3x^2 + 1)^3 (x^2 + 2x) dx = \frac{(x^3 + 3x^2 + 1)^{3+1}}{3+1} \cdot \frac{(x^2 + 2x)}{3(x^2 + 2x)}$$

$$\frac{(x^3 + 3x^2 + 1)^4}{4} \cdot \frac{1}{3}$$

$$\frac{1}{12}(x^3 + 3x^2 + 1)^4$$

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Example 2: $\int (x^2+6x)^2(x+3)dx$

Solution:

$f(x)=(x^2+6x); g(x)=(x+3)$

$f'(x)=2x+6=2(x+3)$

$$\int (x^2+6x)^2(x+3)dx=\frac{\left[x^2+6x\right]^3}{3}.\frac{(x+3)}{2(x+3)}$$

$$=\frac{1}{6}\left[x^2+6x\right]^3+C$$

The type of relationship that must exist between the two functions in this case must be that the derivative of the higher polynomial function is perfectly divisible by the lower polynomial function. It may in some cases involve differentiating the higher polynomial first to know if it gives the multiple of what we refer to as the lower polynomial.

Integral of a sum and multiple

For an integral which has the function as a sum of separate parts in the explanatory variable, the whole integration is separable into individual units.

If $f(x)=g(x)+h(x)$,

then $\int f(x)dx=\int\left[g(x)+h(x)\right]dx=\int g(x)dx+\int h(x)dx+C$

Example 1: $f(x)=3x^3+4x+5$, Find $\int f(x)dx$

$$\int f(x)dx=\int\left[3x^3+4x+5\right]dx$$

$$=\int 3x^3dx+\int 4xdx+\int 5dx+C$$

$$=\frac{3x^4}{4}+2x^2+5x+C$$

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Example 2: If $f(x) = 2x^3 + 5x^2 + x + 5$, Find $\int f(x)dx$

Solution

$$\int f(x)dx = \int (2x^3 + 5x^2 + x + 5)$$

$$= \int 2x^3 dx + \int 5x^2 dx + \int x dx + \int 5 dx$$

$$= \frac{1}{2}x^4 + \frac{5}{3}x^3 + \frac{1}{2}x^2 + 5x + C$$

The integral of a multiple may either be solved by using integration by substitution method or solving through the use of integration by part – this is particularly useful if none of the functions is the derivative of the other.

Integration by Substitution

Example 1: Solve $\int 12x^2(x^3 + 2)dx$

Solution:

Assume $u = x^3 + 2$, such that we have $\int 12x^2 + udx$

Take the derivative of u.

$\frac{du}{dx} = 3x^2$; make dx the subject of the formula

$dx = \frac{du}{3x^2}$; then

$$\int 12x^2 + udx = \int 12x^2 .u \frac{du}{3x^2}$$

$$\int 4udu = \frac{4u^2}{2} + C = 2u^2 + C$$

Then, substitute $u = x^3 + 2$ back into the answer

$$f(x) = 2(x^3 + 2)^2 + C$$

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Example 2: $\int 4x^3(2x^4+10)dx$

Let $u = 2x^4+10$

$$\frac{du}{dx} = 8x^3$$

Make dx the subject of the formula

$$dx = \frac{du}{8x^3}$$

$$\int 4x^3(2x^4+10)dx = \int 4x^3(u)dx$$

Substitute $dx = \frac{du}{8x^3}$

$$\int 4x^3 u.\frac{du}{8x^3}$$

$$\frac{1}{2}u.du$$

$$\int = \frac{1}{2}\int u.du = \frac{1}{2}.\frac{u^2}{2}+C$$

$$= \frac{u^2}{4}+C$$

Substitute $u = 2x^4+10$ back into the answer

$$\frac{1}{4}(2x^4+10)^2+C$$

Example 3: $\int 9x^2(x^3+2)^8 dx$

Let $u = x^3+2$; and $\frac{du}{dx} = 3x^2$

Make dx the subject of the formula; $dx = \frac{du}{3x^2}$

$$\int 9x^2(x^3+2)^8 dx = \int 9x^2.u^8 dx$$

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$$= \int 9x^2 . u^8 \frac{du}{3x^2}$$

$$= \int 3u^8 du$$

$$= \frac{3u^9}{9} + C = \frac{1}{3}u^9 + C$$

Substituting $u = x^3 + 2$

$$\frac{1}{3}(x^3 + 2)^9 + C$$

In all the above examples, it can be seen that integration by substitution can be approached through the use of the following steps.

If $y = f(x)g(x)$

$$\int ydx = \int [f(x)g(x)].dx$$

Assuming that the polynomial g(x) is of a higher degree relating to f(x), then the steps to be taken are:

a. let the function with the higher order be expressed as another function say $u = g(x)$

b. take the derivative of u with respect to x. i.e. $\frac{du}{dx}$

c. solve algebraically for dx

d. substitute u for g(x) and substitute the value of dx for it. $\int ydx = \int f(x).u \frac{du}{g'(x)}$ such that $f(x)$ and $g'(x)$ can divide each other.

e. Integrate with respect to u

f. Substitute g(x) in place of u in the final answer; such that the final answer reflects a function of x as the solution.

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Integration by Part

Integration by part is derived through the corresponding integral rule for product rule. That is, this is derived as the converse of the product rule in differentiation.

$$\frac{d}{dx}\left[u(x).v(x)\right] = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

Integrating both sides with respect to dx we have

$$\int \frac{d}{dx}\left[u(x).v(x)\right]dx = \int u(x)\frac{dv}{dx} + \int v(x)\frac{du}{dx}$$

$$u(x).v(x) = \int u(x)\frac{dv}{dx} + \int v(x)\frac{du}{dx}$$

Thus, the formula for the integration by part is

$$\int u(x)\frac{dv}{dx} = u(x).v(x) - \int v(x)\frac{du}{dx}$$

OR

$$\int v(x)\frac{du}{dx} = u(x).v(x) - \int u(x)\frac{dv}{dx}$$

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LECTURE THIRTEEN

Economic Application of Integration

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The usefulness of integral calculus in economics cannot be overemphasized. Some of its applications are presented in this section.

Recall that the derivative of a total function gives its marginal function. For example, the derivative of the total revenue function yields the marginal revenue function $MR = \frac{dTR}{dq}$. You can reverse the operation by taking the integral of the marginal function to obtain the total function. For example, $\int MRdq = \int \frac{dTR}{dq}.dq + K$

Example 1: For a marginal cost function $MC(q) = 2q + 1$. Find the total cost function at q = 10, assuming that the fixed cost is equal to 10.

Solution:

$TC(q) = \int MCdq = \int 2q + 1 = \frac{2}{2}q^2 + q + C$

TC = $q^2 + q + C$

When q = 0, TC = FC

$(0)^2 + 0 + C = 10$

$C = 10$

Therefore, the total cost function is $TC(q) = q^2 + q + 10$

Specifically, total cost when q = 10 is obtained as follows.

$TC = 10^2 + 10 + 10$
$= 100 + 10 + 10 = 120$

Example 2: If $MP_L = 5L^{1/3}$ is a firm's marginal product, given that labour is the only factor of production, find the firm's production function.

Solution:

$$Q = \int MP_L dq$$

$$Q = \int 5L^{1/3} dl = \frac{3}{4}.5L^{3/4} + C$$

$$Q = \frac{15}{4} L^{3/4} + C$$

Q = 0 when L = 0
Then ;

$$0 = \frac{15}{4}(0)^{3/4} + C$$

$$C = 0$$

Therefore, $Q = \frac{15}{4} L^{3/4}$

Definite Integral

Definite integrals are integrals with limit/boundary. The introduction of boundary helps to eliminate the arbitrary constant.

Example 1: $\int_0^2 (5x + 3)dx$

$$= \left[\frac{5x^2}{2} + 3x \right]_0^2$$

$$= \left[\frac{5(2)^2}{2} + 3(2) \right] - \left[\frac{5(0)^2}{2} + 3(0) \right]$$

$$= [10 + 6] - 0$$

$$= 16$$

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Example 2: For a firm with the marginal revenue $MR = 85 - 5Q$, what is the change in total revenue if it expands its output from 2 to 6?

Solution:

$$\int_2^6 (85 - 5q)dq = \left[85q - \frac{5q^2}{2}\right]_2^6$$

$$= \left[85(6) - \frac{5(6)^2}{2}\right] - \left[85(2) - \frac{5(2)^2}{2}\right]$$

$$= 420 - 160$$

$$= 260$$

The solution of definite integral may sometimes involve sketching the graph of the integral. This is only possible where a full polynomial is given.

Measuring changes in capital stock also requires the use of definite integral.

Example 3: The rate of net investment is $I = 40t^{3/5}$ and capital stock at t = 0 is 75. Find the capital function.

Solution:

$$K = \int Idt = \int 40t^{3/5}dt = 40\left(\frac{5}{8}t^{8/5}\right) + C$$

$$K = 25t^{8/5} + C$$

When t = 0, K = 75.

Then $25(0)^{8/5} + C = 75$

$$C = 75$$

Therefore, $K = 25t^{8/5} + C$; and

$$K = 25t^{8/5} + 75$$

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Example 4: The rate of net investment $I = 60t^{1/3}$ and capital stock at t = 1 is 85. Find K.

Solution:

$$K = \int I dt = \int 60t^{1/3} + C = 45t^{4/3} + C$$

When t = 1; K = 85

$$45(1)^{4/3} + C = 85$$

$$C = 85 - 45 = 40$$

Therefore, $K = 45t^{4/3} + 40$

Consumers' and Producers' Surplus

A demand function $P_1 = f_1(Q)$, fig. 1, represents the different prices consumers are willing to pay for different quantities of a good. If equilibrium in the market is at (Q_0, P_0), then the consumers who would be willing to pay more than P_0 benefit.

Total benefit to consumers is represented by the shaded area and is called consumers' surplus.

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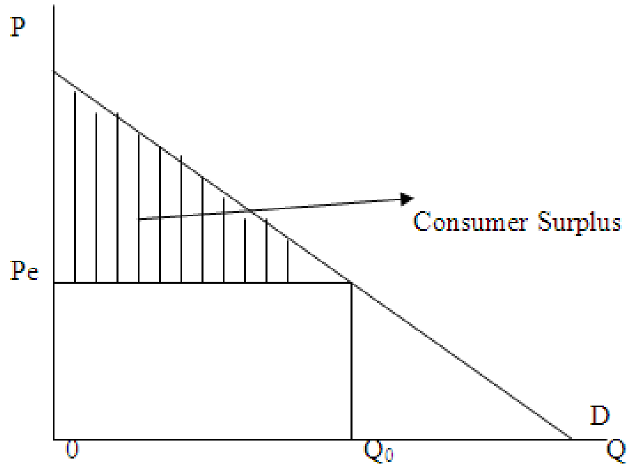


Fig. 2

Mathematically,

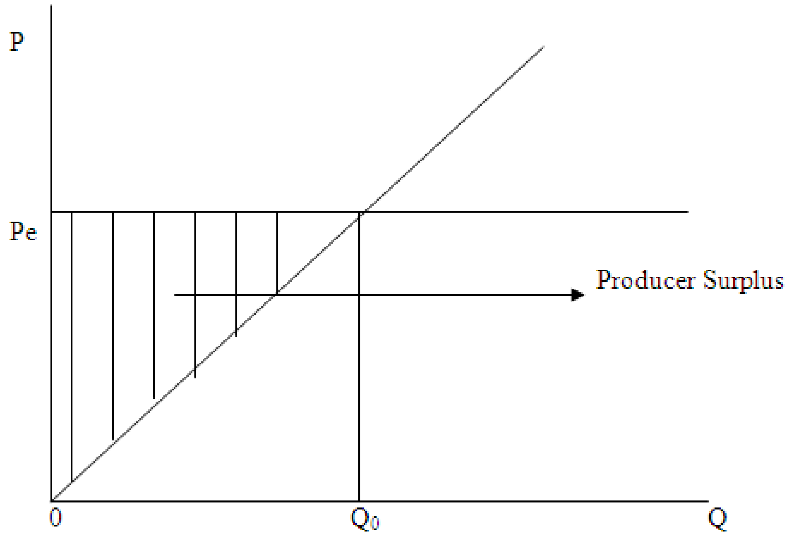
Consumers' surplus $= \int_0^{Q_0} f_1(Q)dQ - Q_0P_0$

A supply function $P_2 = f_2(Q)$, fig, 2, represents the different prices at which quantities of a good will be supplied. If market equilibrium occurs at $(Q_{0,}P_0)$, the producers who would supply at a lower price than P_0 benefit.

Total gain to producers is called producers' surplus and is designated by the shaded area.

Mathematically,

Producers' surplus $= Q_0P_0 - \int_0^{Q_0} f_2(Q)dQ$



Example: Given the demand function $P = 42 - 5Q - Q^2$, assuming that the equilibrium price is 6. What is the consumer surplus?

Solution:

At $P_0 = 6$, the demand function becomes

$$42 - 5Q - Q^2 = 6$$
$$36 - 5Q - Q^2 = 0$$
$$(Q + 9)(-Q + 4) = 0$$
$$Q = -9, Q = 4$$

Negative quantity is not feasible, so we neglect – 9.

So $Q_0 = 4$

Consumer surplus $= \int_0^4 (42 - 5Q - Q^2)dQ - 4(6)$

$$\left[42Q - \frac{5Q^2}{2} - \frac{Q^3}{3}\right]_0^4 - 24$$
$$= 82\, {}^{2}\!/_{3}$$

Practice Exercise

1. Solve the following

   a. $\int x\sqrt{3x^2 + 5}dx$

   b. $\int \frac{x+1}{x^2 + 2x + 5}dx$

   c. $\int (6x^5 + 4x^4 + 2x^3 + 5)^4 (15x^4 + 8x^3 + 3x^2)dx$

   d. $\int \left(\frac{3x^2 + 2}{x^3 + 2x + 1}\right)dx$

2. Find an expression for total cost (TC) if the fixed cost (FC) and marginal cost (MC) are 92 and $3Q^2 - 28Q + 84$ respectively.
3. If the marginal revenue is given by $MR = 120 - 8Q$, find an expression for total revenue.
4. Integrate the following functions for x varying between 0 and 2

   a. $6x^{1/3}$

   b. $18x^3 - 3x + 4$

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5. Determine the total cost function given that fixed cost is 125 and the marginal cost $MC = 5Q^2 - 22Q + 56$

6. Given $I(t) = 9t^{1/2}$, find the level of capital formation
   a. in 8 years [0,8]
   b. from fourth year through eight year [4,8]

7. A polynomial which can be written as $(Q+1)(Q+1)(Q-2)-3 = P$ is given as the demand function. Calculate the consumer supply if the equilibrium price $P_0 = 3$

8. If the marginal propensity to consume (MPS) is given as $S'(Y) = 0.3 - 0.1Y^{-1/2}$ and if the aggregate savings is nil when $Y = 81$. Find the savings function S(Y).

9. Suppose that the net investment flow is described by the equation $I(t) = 3t^{1/2}$ and that the initial capital stock at time t = 0 is K (0), what is the time path of capital K.

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