BASIC PRINCIPLE OF PHYSICS II (INTRODUCTORY ELECTRICITY AND MAGNETISM)

A COURSE MATERIAL FOR PHY 104

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General Introduction and Course Objective

Course Curriculum Contents

Coulomb's law, electric charges and methods of charging. Electric field intensity and charge distribution in conductors and insulators of various configurations. Electric potential, potential gradient and the electrical potential energy. Capacitors and dielectric. Ohm's law and analysis of direct-current circuits containing only resistors, cells and simple circuit laws e.g. Kirchhoff's laws. The Wheatstone bridge and potentiometer and their applications. Electrodynamics of charged particles, Magnetic fields and magnetic forces of/on current-carrying conductors. Applications to measuring instruments. Concept of Electromagnetic induction and applications: motors, dynamos, generators, etc. A.C voltages applied to Inductors, Capacitors and resistors singly and combined. The transformers.

3 Units, Compulsory

Table of Contents

Study Session 1: Electric charges, Methods of charging and Coulomb's law

Expected Duration: 1 week or 2 contact hours

Introduction

The existence of electric charge both as static charge and electric current, caused by moving electric charges, are manifested in many ways. Lightning that is seen in the sky during thunderstorm is an example of the accumulation of static electric charge in clouds. The deflection of electron beam in your computer display, and the flow of electric current in conductors all are examples of moving charges.

The effects of electric charges are experienced in many circumstances. A spark is seen when we try to remove our sweater or synthetic clothes from our body, particularly during dry weather. A sensation of an electric shock is experienced while opening the door of a car or holding the iron bar of a bus after sliding from our seat. The reason for these experiences is discharge of electric charges through our body, which were accumulated due to rubbing of insulated surfaces. In this session, you will learn about the two kinds of electric charges, their behavior in different circumstances and the forces that act between them.

Learning Outcomes

When you have studied this session, you should be able to explain:

1.6 The two types of electric charges
1.7 Properties of electric charges.
1.8 Methods of charging.
1.9 Classification of materials.
1.10 Coulomb's law of force between electric charges and the principle of superposition.

1.1 Types of Electric Charges

All matters are made up of atoms which consists of electrons, protons and neutrons. Electric charge is an electrical property of the atomic particles (electrons, protons and neutrons) of a matter measured in Coulombs (C). The charge 'e 'on an electron is negative and is equal in magnitude to 1.6×10^{-19} C. The symbol "e" represents only the magnitude of the charge on a proton or an electron and does not include the algebraic sign that indicates whether the charge is positive or negative. A proton carries a positive charge of the same magnitude as the electron. In a neutral atom, there are equal numbers of protons and electrons which makes the atom to be neutrally charged. Many experiments conducted by Benjamin Franklin revealed that there are two kinds of electric charges: positive and negative charges. A glass rod that has been rubbed with silk is commonly used as an example of identifying positive and negative charges. Another example is a hard rubber that has been rubbed with fur. When two objects are rubbed together, electrons can be transferred from one object to the other; the protons that are tightly bound in nuclei are not transferred. Due to the nature of atoms or molecules making up a substance, it may accept electrons or give up electrons when rubbed with another substance. For example, when silk cloth is rubbed against glass rod, electrons are removed from the glass rod and deposited on the silk. The silk gains electrons from glass because it has a greater affinity for electrons. Hence, silk having excess electrons is negatively charged, and glass having a deficiency of electrons is positively charged. Similarly, when a rubber rod is rubbed against fur, electrons are removed from the fur and deposited on the rubber rod. The rubber rod is said to be negatively charged because an excess of electrons. The fur is said to be positively charged because of deficiency of electrons.

1.2 Properties of Electric Charges

Many experiments can be used to explain the basic properties of electric charge which is considered to be one of the fundamental properties of matter. When two negatively charged rubber rods are brought close to each other, the two rods repel each other, as shown in fig. 1.1a. Conversely, if a positively charged glass rod is brought near a suspended negatively charged rubber rod, the two rods attract each other, as shown in Fig.1.1b. Based on these observations, it can be concluded that there are two kinds of charges in nature; one is positive and the other is negative, and they obey the following properties:

Like charges repel each other and unlike charges attract each other.

Additionally, it was found that when one object is rubbed with another, charge is transferred between them, that is, charge is not created in the rubbing process. Therefore,

The total electric charge in any isolated system is conserved or electric charge can neither be created nor destroyed, but it can only be transferred.

In 1909, Robert Milikan discovered that an electric charge always occurs in integral multiples of a fundamental unit of charge, which is taken as the charge on an electron. In a modern view, the electric charge q is said to be **quantized** and we can write $q = ne$, where n is an integer (n = ±1, ±2, ±3 ----) and e is charge on an electron. It means that a charged body cannot have 1.5e or 4.3e. Therefore,

Charge is quantized.

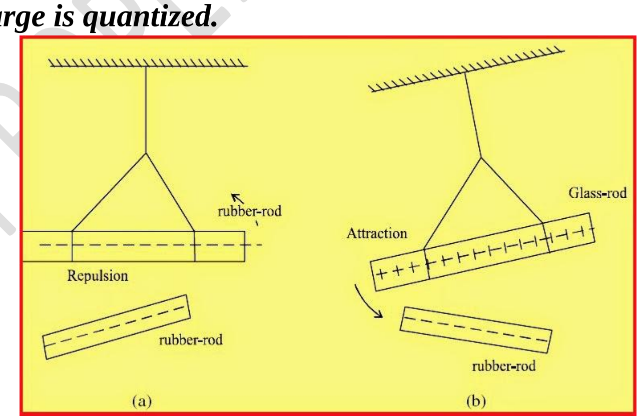


Figure 1.1 : (a) A negatively charged rubber rod is repelled by another negatively charged rubber rod. (b) A negatively charged rubber rod suspended by a thread is attracted to a positively charged glass rod.

1.3 Methods of Charging

The process of supplying the electric charge (electrons) to an object or losing the electric charge from an object is called charging. There are three methods by which an uncharged object can be charged. The three methods are: charging by conduction, charging by friction and charging by induction.

1.3.1 Charging by conduction

The process of charging an uncharged object by bringing it in contact with another charged object is called charging by conduction. A charged object has unequal number of electrons and protons. Hence, when a charged object is brought in contact with the uncharged conductor, the electrons are transferred from the charged object to the conductor. This method is applicable for charging metals and other conductors. For instance, consider an uncharged (neutral) metal sphere resting on an insulating stand and a positively charged aluminum plate brought in contact with it. Suppose the uncharged metal sphere is now touched with the positively charged aluminum plate, there is transfer of electrons from the metal sphere to the positively charged aluminum plate. Hence, uncharged metal sphere loses electrons and charged aluminum plate gains electrons. Thus, uncharged metal sphere becomes positively charged by losing extra electrons (see fig.1.2 (i-iii) for the illustration).Similarly, uncharged conductor becomes negatively charged if it is brought in contact with a negatively charged conductor.

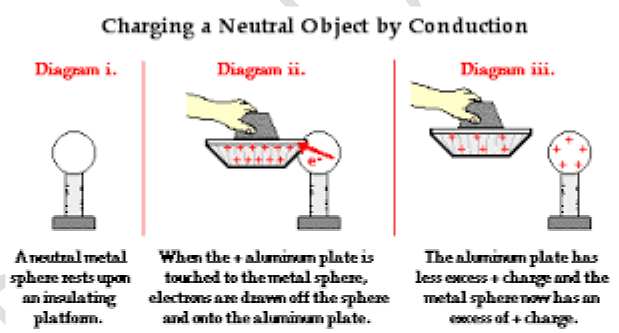


Figure 1.2: charging a neutral object by conduction

1.3.2 Charging by friction

When an object is rubbed over another object, the electrons are transferred from one object to another. This transfer of electrons takes place due to friction between the two objects. The object that transfer electrons loses negative charge (electrons) and the object that accepts electrons gains the negative charge (electrons). Hence, the object that gains extra electrons becomes negatively charged and the object that loses electrons becomes positively charged. Thus, the two objects get charged by friction. This method of charging an object is called charging by friction. For example, when a glass rod is rubbed with silk, it will develop a positive charge (See fig. 1.3). This is because frictional forces between the silk and the glass remove electrons from the glass rod and deposit them on the silk.

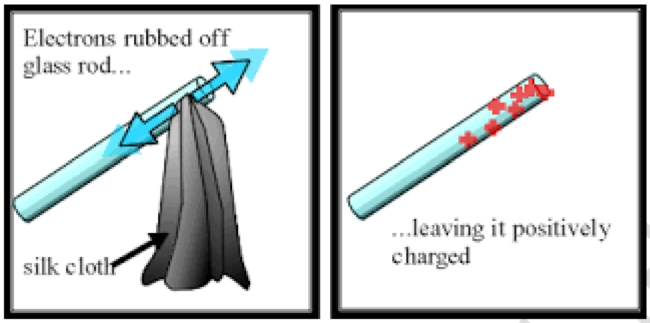


Figure 1.3: Charging by friction (When a neutral glass rod is rubbed with a neutral silk cloth, electrons are transferred from the glass rod to the silk cloth, leaving the glass rod positively charged)

1.3.3 Charging by induction

The process of charging an uncharged object by bringing another charged object near it, but not touching it, is called charging by induction. For example, consider an uncharged metal sphere and a negatively charged plastic rod a shown below in (diagram 1). If the negatively charged rod is brought near to an uncharged metal sphere as shown below in (diagram 2), charge separation occurs. The positive charges in the sphere get attracted towards the plastic rod and move to one end of the sphere that is closer to the plastic rod. Similarly, negative charges are repelled from the plastic rod and move to another end of the sphere that is farther away from the plastic rod. Thus, the charges in the sphere rearrange themselves in a way that all the positive charges are nearer to the plastic rod and all the negative charges are farther away from it. If this sphere is connected to a ground through the wire as shown in (diagram 3), free electrons of the sphere at farther end flow to the ground. Thus, the sphere becomes positively charged by induction. If the plastic rod is removed as shown in (diagram 4) all the positive charges spread uniformly in the sphere.

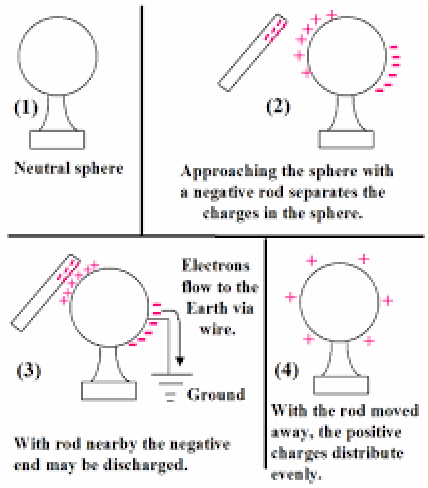


Figure 1.4: Charging by induction

1.4 Classification of Materials

Materials vary in their ability to conduct electric charges, and this ability is determined by how tightly or loosely the electrons are held to the nucleus. Therefore, materials can be classified according to their ability to conduct electric charge. The materials can be classified as: conductors, insulators, semiconductors and superconductors.

1.4.1 Conductors

Conductors are materials containing some electrons that can move freely. In conductors, the electrons furthest away from the nucleus in the outer shell (valence electrons) are not strongly attracted by the nucleus of the atom. Therefore, the electrons in conductors can move around freely. Metals have millions of free electrons that can take part in the conduction of electric charge. Examples of conductors include: metals (silver, copper, gold, aluminum, iron, etc.), salt solution, acids, graphite, water, wet wood and the human body. When such materials are charged by rubbing in some small region, the charge readily distributes itself over the entire surface of the material.

1.4.2 Insulators

Materials such as glass, rubber, dry wood, diamond, plastics, paper, oil, ceramic, etc. fall into the category of electrical insulators or nonconductors. Electrical insulators are materials in which all electrons are bound are bound to atoms and cannot move freely through the materials. When such materials are charged by rubbing, only the area rubbed becomes charged and the charged particles are unable to move to other regions of the material.

1.4.3 Semiconductors

Semiconductors are materials that are intermediate between a good conductor and a good insulator. Examples of semiconductors are silicon, gallium, arsenide and germanium. The electrical properties of semiconductors can be changed drastically by adding certain amount of impurities element into their crystal structure. Some semiconductors act like insulators at low temperatures, while at room temperatures and above they act as conductors. For example, selenium (which is used on the drums of some photocopiers) depends on the amount of light it is exposed to. It is an insulator in the dark, but becomes a conductor when exposed to light. Generally, the conductivity of semiconductor increases with increasing temperature, in contrast to metallic conductor. Semiconductors have widespread application in the electronic industry due to their ability to conduct or insulate in different situations.

1.4.4 Superconductors

A superconductor is a material that conducts electricity without resistance below a certain temperature. Superconductors usually work only at very cold temperatures near absolute zero. Absolute zero is the temperature at which atoms have no kinetic energy and stop moving. This happens at -273.15^0 which is 0 K. Some ceramic based superconductors have been created that work at around the same temperatures as liquid nitrogen (about -200^0 C) which is very easy and cheap to make. Examples of superconductors are mercury, lead, niobium, niobium-titanium, germanium-niobium and niobium nitride.

1.5 .1 Coulomb's law

Charles Coulomb (1736- 1806) studied the force exerted by one charge on another charge using a torsional balance. In Coulomb's experiment, the charged sphere were much smaller than the distance between them so that the charges could be treated as point charge. From measurements made on the quantity of charge, the distance between the charges and the forces acting on the charges, Coulomb was able to show that the:

(a) magnitude of the force was proportional to the product of the charges

(b) magnitude of the force was inversely proportional to the square of the distance separating the charges

(c) direction of the force was along a line joining the centres of the charges

(d) force between the charges is either attractive or repulsive, depending on the nature of the charges. The force is repulsive if the charges have the same sign and attractive if the charges have opposite sign.

The magnitude of the electric force F between charges q_1 and q_2 separated by a distance r is given by:

$$F = \frac{k q_1 q_2}{r^2} \tag{1.1}$$

Where k is the constant of proportionality, $k = \frac{1}{4\pi\varepsilon_0}$ for free space or vacuum and $k = \frac{1}{4\pi\varepsilon}$ for a material medium. ε_0 is called the permittivity of free space and ε is the permittivity of the medium. It means that if the same system of charges is kept in a material medium, the magnitude of the force will be different from that in free space.

$$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \tag{1.2}$$

$$\left(k = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\ \frac{N\,m^2}{C^2}\right) \tag{1.3}$$

1.5.2 Principle of Superposition

Coulomb's law is applicable to any pair of point charges. When more than two charges are present, the net force on any one charge is simply the vector sum of forces exerted on it by the other charges. This is the principle of superposition. For example, if three charges are present, the resultant force experienced by q_3 due to q_1 and q_2 will be

$$\mathbf{F}_3 = \mathbf{F}_{13} + \mathbf{F}_{23} \tag{1.4}$$

Worked Examples

1.1 Common static electricity involves charges ranging from nanocoulombs to microcoulombs.
   (a) How many electrons are needed to form a charge of -2.0 nC ?
   (b) How many electrons must be removed from a neutral object to leave a net charge of 0.50 µC?
   Solution:
   (a) One electron has a charge q_e = -1.6 x 10^{-19} C. The number of electrons needed to generate a total charge of -2.0 nC is:

$$N = \frac{Q}{q_e} = \frac{-2.0 \times 10^{-9}\ C}{-1.6 \times 10^{-19}\ C} = 1.25 \times 10^{10} \text{ electrons}$$

(b) Number of electrons to be removed from a neutral object is:

$$N = \frac{-5.0 \times 10^{-6}\ C}{-1.6 \times 10^{-19}\ C} = 3.13 \times 10^{12} \text{ electrons}$$

1.2 Consider the arrangement of charges shown in the figure1.5 below. Calculate the net force on charge A in each configuration if the distances are r_1 = 12.0 cm and r_2 = 20.0 cm.

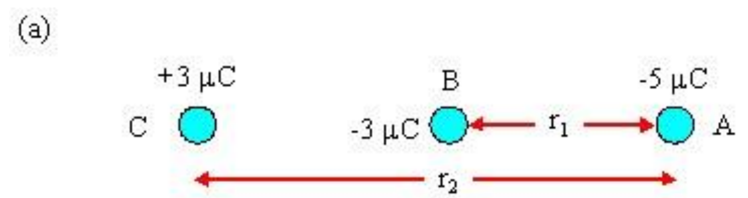


Figure 1.5

Solution:

Charge A is the target and charges B and C are sources. Charge B and A have the same sign, so they repel. That is, charge A feels a force F_{BA} directed in the +i direction. Charge C and A have the opposite sign, so they attract. That is, charge A feels a force F_{CA} directed in the –i direction. The situation is shown in the fig. 1.6 below:

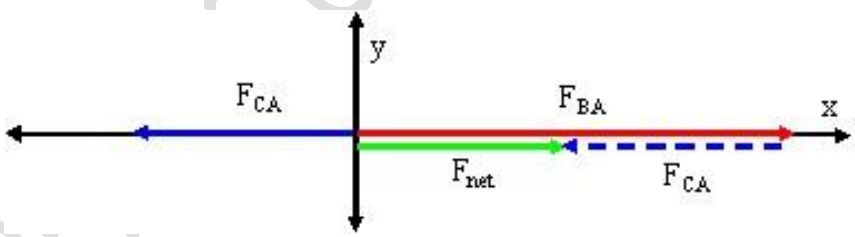


Figure 1.6

Since we know the exact direction of these forces we need only calculate the magnitude of these forces:

$$F_{BA} = k\left|\frac{Q_B\,Q_A}{r_1^2}\right| = \left(9.00 \times 10^9\ N.m^2\,C^{-2}\right)\left|\frac{\left(-3 \times 10^{-6}\ C\right)\left(-5 \times 10^{-6}\ C\right)}{(0.12\ m)^2}\right| = 9.375\ N$$

and

$$F_{CA} = k\left|\frac{Q_C\,Q_A}{r_2^2}\right| = \left(9.00\times10^9\ N.m^2\,C^{-2}\right)\left|\frac{\left(+3\times10^{-6}\ C\right)\left(-5\times10^{-6}\ C\right)}{(0.20\,m)^2}\right| = 3.375\ N$$

In vector components form, the two forces are: $F_{BA} = i[9.375\ N]$ and $F_{CA} = -i[3.375\ N]$

The net force, $F_{net} = F_{BA} + F_{CA} = i(6.00\ N)$

The net force acting on charge A due to the other forces is 6.00 N along the positive x axis.

(b)

Figure 1.7

The magnitudes of F_{BA} and F_{CA} are the same as in part (a), however, the direction of the force of charge C on charge A is different. That force is now F_{CA} = +i 3.37125 N directed to the right as shown in the diagram below:

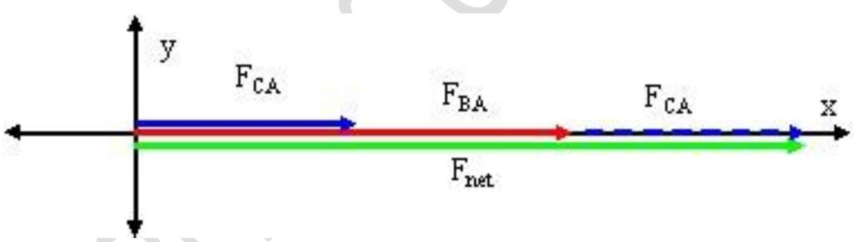


Figure 1.8

Since the forces are along the same axis, the net force is given by:

$$F_{net} = F_{BA} + F_{CA} = i(9.375\ N) + i(3.375\ N) = i(12.75\ N)$$

The net force acting on charge A due to the other forces is 12.75 N along the positive x axis.

1.3 A positive charge of +1 μC is placed between charges of 5.0 μC and 2.0 μC as shown in the diagram below. At what point would the net electrostatic force on it is zero?

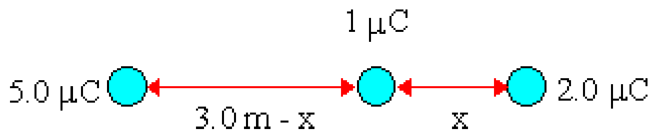


Figure 1.9

Solution:

The charge 1 μC is placed where the force from the right 2.0 μC charge is cancelled by the force from the left 5.0 μC charge. Let's assume that the 1 μC charge is some distance x from the 2.0 μC.

$$F_{51} = F_{21}$$

$$k\frac{q_5\, q_1}{(3-x)^2} = k\frac{q_2\, q_1}{x^2}$$

By eliminating the common factor $k\, q_1$ gives: $\dfrac{q_2}{x^2} = \dfrac{q_5}{(3-x)^2}$

Taking the square root on both sides and collecting the like term gives:

$$x = 3\frac{\sqrt{q_2}}{\sqrt{q_2} + \sqrt{q_5}} = 3\frac{\sqrt{2}}{\sqrt{2} + \sqrt{5}}$$

$$x = 1.162\, m$$

The 1 μC charge must be placed at 1.162 m from the 2 μC charge for it to experience no net force.

1.4 A certain charge Q is divided into two parts q and Q – q, which are then separated by a certain distance. What must q be in terms of Q in order to maximize the electrostatic repulsion between the two charges?

Solution:

If the distance between the two (new) charges is *r*, then the magnitude of the forces between them is:

$$F = k\frac{(Q-q)q}{r^2} = \frac{k}{r^2}\left(Qq - q^2\right)$$

Since the force between the charges is repulsive, the charges Q and Q – q both have the same sign and q(Q – q) will yield a positive number. In order to determine the value of q which gives maximum F, we take the derivative of F with respect to q and find where it is zero:

$$\frac{dF}{dq} = \frac{k}{r^2}(Q-2q) = 0$$

which has the solution $(Q-2q) = 0 \;\Rightarrow q = \frac{Q}{2}$

So the maximum repulsive force is obtained by dividing the original charge *Q* in half.

1.5 A neutron consists of one "up" quark of charge $+\frac{2e}{3}$ and two " down" quarks each having charge $-\frac{e}{3}$. If the down quarks are 2.6 x 10^{-15} m apart inside the neutron, what is the magnitude of the electrostatic force between them?

Solution:

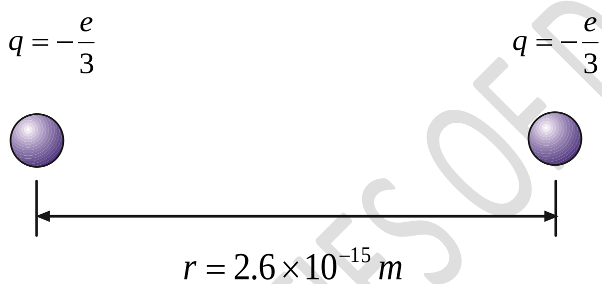


Figure 1.10

We picture the two down quarks as shown in the above figure. We use Coulomb's law to find the force between them. (It is *repulsive* since the quarks have the same charge.) The two charges are:

$$q_1 = q_2 = -\frac{e}{3} = -\frac{\left(1.60\times10^{-19}\ C\right)}{3} = -5.33\times10^{-20}\ C$$

and the separation is $r = 2.6 \times 10^{-15}$ m. The magnitude of the force is:

$$F = k\frac{|q_1||q_2|}{r^2} = \frac{\left(9.00\times10^{9}\ N.m^2\,C^{-2}\right)\left(5.33\times10^{-20}\,C\right)\left(5.33\times10^{-20}\ C\right)}{\left(2.6\times10^{-15}\,m\right)^2}$$

$$= 3.8\ N$$

1.6 Two point charges; $Q_A = +0.6\,\mu C$ and $Q_B = -0.2\,\mu C$ are separated by a distance of 5.0 cm.

Find: (a) The electric force acting on each charge.

(b) The electric force acting on each charge if their distance is doubled.

(c) The electric force acting on each charge if charge Q_A is doubled.

Solution:

Figure 1.11

From Coulomb's law, $F_{BA} = k\dfrac{|Q_A||Q_B|}{r^2}$

(a) $F_{AB} = F_{BA} = k\dfrac{|Q_A||Q_B|}{r^2} = \left(9.00\times 10^9\ N\,m^2\,C^{-2}\right)\dfrac{\left(0.6\times 10^{-6}\ C\right)\left(0.2\times 10^{-6}\ C\right)}{(0.05\,m)^2} = 0.43\,N$

(b) When the distance is doubled $r = 5.0\,cm \times 2 = 10.0\,cm$

$$F_{AB} = F_{BA} = k\frac{|Q_A||Q_B|}{r^2} = \left(9.00\times 10^9\ N.m^2\,C^{-2}\right)\frac{\left(0.6\times 10^{-6}\ C\right)\left(0.2\times 10^{-6}\ C\right)}{(0.10\,m)^2}$$

$= 0.11\,N$

(c) If charge Q_A is doubled, it becomes, $Q_A = 1.2\,\mu C$

$$F_{AB} = F_{BA} = k\frac{|Q_A||Q_B|}{r^2} = \left(9.00\times 10^9\ N.m^2\,C^{-2}\right)\frac{\left(1.2\times 10^{-6}\ C\right)\left(0.2\times 10^{-6}\ C\right)}{(0.05m)^2} = 0.86\ N$$

Self-Assessment Questions (SAQs)

1. (a) Explain the quantization of electric charge of a body.
   (c) Explain why quantization of electric charge can be ignored when dealing with large scale charges.

2. Determine the total charge of 75.0 kg of electron.

3. Two identical objects with equal and opposite charges separated by a distance d exert a force of -2.5 N on each other. Determine the force exerted by the objects if the distance between them becomes 2d.

4. The magnitude of electrostatic force between two points charges $q_1 = 26.0\mu$C and $q_2 = -47.0\mu$C is 5.70 N. Calculate the distance between the two charges.

5. Two small positively charged spheres have a combined charge of 5.0×10^{-5} C. If each sphere is repelled from each other by an electrostatic force of 1.0 N when the spheres are 2.0 m apart, what is the charge on each sphere?

6. Four point charges $q_A = 2\,\mu C$, $q_B = -5\,\mu C$, $q_C = 2\,\mu C$ and $q_D = -5\,\mu C$ are located at the sides of a square which is of 10 cm side shown below. Calculate the force acting on a charge of $1\,\mu C$ when placed at the centre of the square.

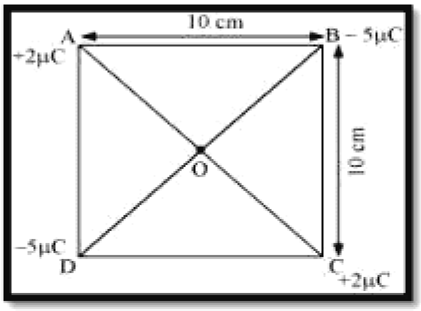


Figure 1.12

7. In a fission , a nucleus of uranium -238, which contains 92 protons, divides into two smaller spheres, each having 46 protons and a radius of 5.9 x 10^{-15} m. Calculate the magnitude of the repulsive electric force pushing the two spheres apart.

8. Two identical conducting spheres, fixed in place, attract each other with an electrostatic force of 0.108 N when separated by 50.0 cm, centre-to-centre. The spheres are then connected by a thin conducting wire. When the wire is removed, the sphere repel each other with an electrostatic force of 0.360 N. What were the initial charges on the spheres?

SUMMARY

In this session, you have learnt that:

1. Electric charges is an electrical property of atomic particles of matter measured in Coulombs. Charges are of two types: positive and negative charges. Electric charge is responsible for the electric force. The force between two like charges is repulsive and is attractive between two unlike charges.
2. Charge is conserved that is the total charge in any isolated system always remain constant. Therefore, electric charge can neither be created nor destroyed but it can only be transferred.
3. Charge is quantized, that is, the magnitude of all charges found in nature are an integral multiple of a fundamental charge *e* , $q = ne$
4. There are three methods by which an uncharged object can be charged. The three methods are: charging by conduction, charging by friction and charging by induction.
5. Materials can be classified according to their ability to conduct electric charge. Materials can be classified as: conductors, insulators, semiconductors and superconductors.

6. The electric force between two stationary point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. The force acts along the line joining the two charges.

$$F = \frac{k q_1 q_2}{r^2} \quad \text{where} \quad \left(k = \frac{1}{4\pi\varepsilon_0} = 9\times 10^9 \ \frac{N\,m^2}{C^2} \right)$$

7. If there are more than two charges (q_1 , q_2 , q_3 ,......., q_n) in an electrostatic field, the total force experienced by charges q_1 is the vector sum of the forces on q_1 exerted by the other charges. This is the principle of superposition.

Study Session 2: Electric Field Intensity and Charge Distribution in Conductors and Insulators of Various Configurations

Expected Duration: 1 week or 2 contact hours

Introduction

The electric field is a way of describing how a point charge or a distribution of discrete or continuous charges influences the space around them. The electric field at various points around an electric charge distribution can be analyzed by measuring the force the distribution creates on a *test charge*. The *test charge* is a positive point charge of unit magnitude. It is assumed that the test charge does not disturb the charge distribution or creates a significant field of its own. The electric field E generated by a set of charges can be measured by putting a point charge q at a given position. The test charge will feel an electric force F. The electric field at the location of the point charge is defined as the force F divided by the electric charge q $\left(\overline{E} = \frac{\overline{F}}{q}\right)$.

The definition of the electric field shows that the electric field is a vector field: the electric field at each point has a magnitude and a direction. The direction of the electric field is the direction in which a positive charged placed at that position will move. In this session, the calculation of the electric field generated by various charge distributions will be discussed. In addition, the motion of a charged particle in a uniform electric field will be discussed.

Learning Outcomes

When you have studied this session, you should be able to explain:

2.5 Electric field intensity and its physical significance.
2.6 The properties of electric field lines.
2.7 Electric dipole, dipole moment and formulate the electric field intensity due to an electric dipole.
2.8 Charge density and the electric fields due to different types of continuous charge distributions.
2.9 The motion of a charged particle in electric field

2.1 The Electric Field Intensity and its Physical Significance

The electrostatic force, like the gravitational force, is a force that acts at a distance, even when the objects are not in contact with one another. This means that electrostatic and gravitational forces can act across an empty vacuum, with no matter to carry them. These types of forces are known as **field forces**. Corresponding to the electrostatic force, an **electric field** is said to exist in the region of space surrounding a charged object. **The electric field exerts an electric force on any other charged object within the field.** An electric charge q produces an electric field everywhere. To quantify the strength of the field created by that charge, we can measure the force that a positive 'test charge' q_o experiences at some point. **The electric field or electric**

field intensity $\vec{E}$ produced by a charge q at the location of a small 'test' charge q_o is defined as the electric force $\vec{F}$ exerted by q on q_o divided by the test charge q_o. It is assumed that q_o is infinitesimally small so that the field generated by q_o does not disturb the 'source charges'.

$$\vec{E} = \frac{\vec{F}}{q_0} \qquad \text{or} \quad \vec{F} = q\vec{E} \tag{2.1}$$

From the field theory point of view, it is assumed that the charge q creates an electric field $\vec{E}$ which exerts a force $\vec{F} = q\vec{E}$ on a test charge q_0. The S.I. unit for electric field is Newton per Coulomb (N/C). The direction of the electric field is the same as the direction of the electric force, since the two are related by a scalar. The dimensions of electric field are $MLT^{-3}A^{-1}$.

Using the definition of electric field given in Eq. (2.1) and the Coulomb's law, the electric field at a distance r from a point charge q is given by:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \tag{2.2}$$

The principle of superposition is also applicable to electric force, just as it holds for the electric force. Using the superposition principle, the total electric field due to a group of charges is equal to the vector sum of the electric fields of individual charges:

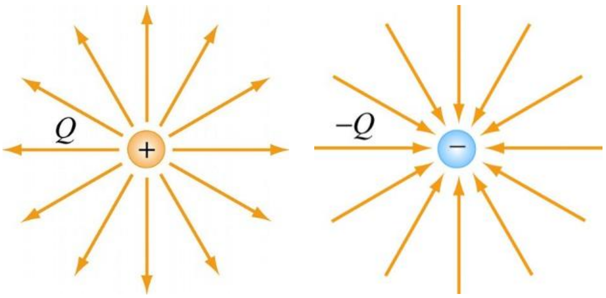
$$\vec{E} = \sum_i \vec{E_i} = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i^2}\hat{r} \tag{2.3}$$

In electrostatics, the electric field at a point in a space around a system of charges can be obtained from the force that a unit positive test charge would experience when placed at that point without disturbing the system. Electric field is a characteristic of the system of charges and is independent of the test charge that is placed at a point to determine the field. The term *field* in Physics generally refers to a quantity that is defined at every point in space and may vary from point to point. Electric field is a vector field, since force is a vector quantity.

The physical significance of the concept of electric field is illustrated when dealing with time dependent electromagnetic phenomena. Suppose we consider the force between two distant charges q_1 , q_2 in accelerated motion. Now the greatest speed with which a signal or information can go from one point to another is c, the speed of light. Thus, the effect of any motion of q_1 on q_2 cannot arise instantaneously. There will be some time delay between the effect (force on q_2) and the cause (motion of q_1). It is precisely here that the notion of electric field (electromagnetic field) is very useful. *The field picture is this: the accelerated motion of charge q_1 produces electromagnetic waves, which then propagate with the speed c, reach q_2 and cause a force on q_2*.The notion of field accounts for the time delay. Although, electric and magnetic fields can be detected only by their effects (forces) on charges, they are regarded as physical entities which have independent dynamics of their own. They can transport energy. The concept of field was first introduced by Faraday and is now among the central concepts in Physics.

2.2 Electric Field Lines

The concept of electric fields lines was introduced by Faraday as an approach to visualize the electric fields. An electric field is an imaginary line drawn in such a way that the direction of its tangent at any point is the same as the direction of the electric field vector. The electric field lines for a positive and a negative charges are shown in Figure 2.1



(a) (b)

Figure 2.1 Field lines for (a) positive and (b) negative charges.

The direction of the electric field lines is radially outward for a positive charge and radially inward for a negative charge. For a pair of charges of equal magnitude but opposite sign (an electric dipole), the field lines are shown in Figure 2.2

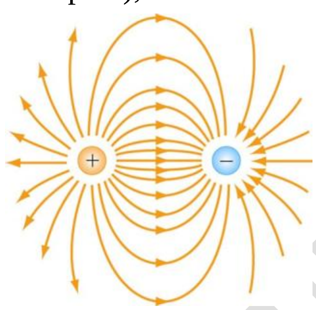


Figure 2.2: Electric field lines for an electric dipole.

The rules for drawing electric field lines are as follows:

- Electric field lines must emerge from a positive charge and end on a negative charge. For a system of charges that has an excess of one type of charge, some lines will emerge or end infinitely far away.
- The number of lines emerging from a positive charge or ending at a negative charge is proportional to the magnitude of the charge.
- Electric field lines cannot cross each other.

An example of electric field lines generated by a charge distribution is shown in Figure 2.3

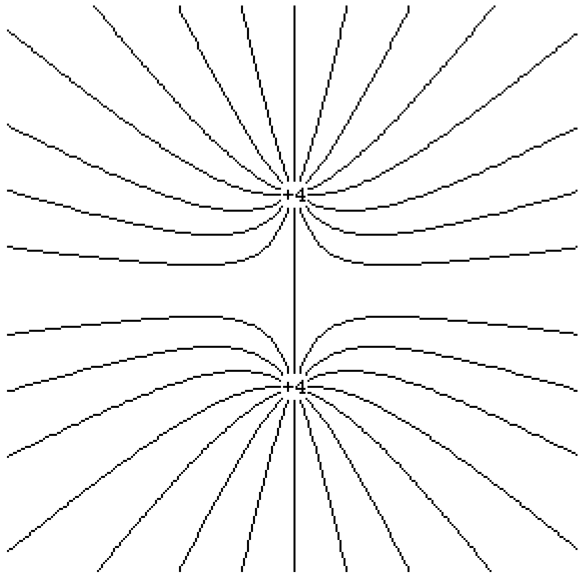


Figure 2.3 :Electric field produced by two point charges q = + 4

The properties of electric field lines may be summarized as follows:

- The direction of the electric field vector **E** at a point is tangent to the electric field lines.
- The number of lines per unit area through a surface perpendicular to the line (density of the lines) is proportional to the magnitude of the electric field in a given region. Thus, the electric field lines are closer together when the electric field is strong, and far apart when the field is weak.
- The electric field lines must emerge from a positive charge (or at infinity) and end on a negative (or at infinity). For a system that has an excess of one type of charge, some lines will emerge or end infinitely far away.
- The number of lines that originate from a positive charge or terminating on a negative charge must be proportional to the magnitude of the charge.
- No two electric field lines can cross each other; otherwise the field would be pointing in two different directions at the same point.

2.3 Electric Dipole

An electric dipole consists of two equal and opposite charges, +q and –q, separated by a distance 2a, as shown in Figure 2.4

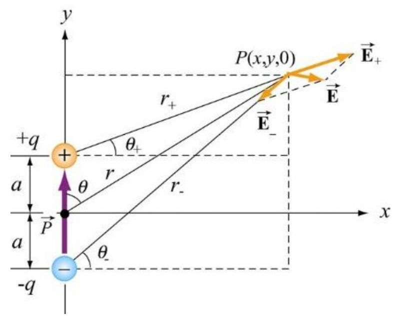


Figure 2.4: Electric dipole

Examples of dipole include CO, H_2O and other polar molecules. The product of the magnitude of charge and separation between the charges is called dipole moment, P:

$$P = q \times 2a \tag{2.4}$$

Its S.I. unit is Coulomb metre. The dipole moment is a vector quantity whose magnitude is given by Eqn. (2.4) and its direction is from negative charge to positive charge along the line joining the two charges (axis of the dipole). The total charge of the dipole is zero but the total electric field of the dipole is not zero because the charges q and –q are separated by some distance. We can now determine the magnitude and direction of electric field due to dipole.

2.3.1 Electric field of a dipole for points on the axis

To derive an expression for the electric field of a dipole at point P which lies on the axis of the dipole, refer to Figure 2.5

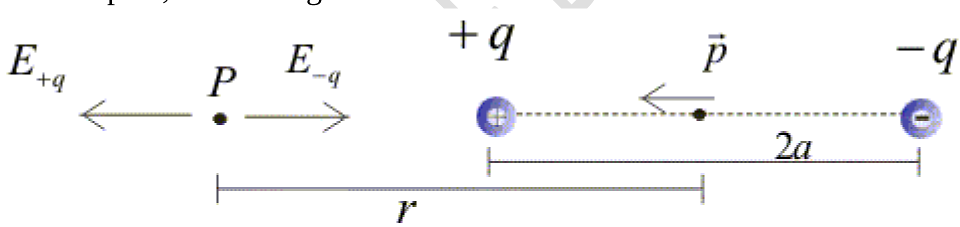


Figure 2.5: Electric dipole for point on axis

Let P be the point at a distance r from the centre of the dipole on side of charge q as shown in Figure 2.5

$$E_{-q} = -q\,\hat{p}\,\frac{1}{4\pi\varepsilon_0 (r+a)^2} \tag{2.5a}$$

$\hat{p} =$ Unit vector along the dipole axis (from –q to +q). Also,

$$E_{+q} = q\,\hat{p}\,\frac{1}{4\pi\varepsilon_0 (r-a)^2} \tag{2.5b}$$

The total field at P is:

$$E = E_{+q} + E_{-q}$$

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right]\hat{p}$$

Or,

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{4ar}{\left(r^2 - a^2\right)}\right]\hat{p} \tag{2.6}$$

For r >> a

$$E = \frac{4qa\,\hat{p}}{4\pi\varepsilon_0\, r^3} \tag{2.7}$$

In terms of electric dipole moment, the electric field of a dipole at large distances becomes:

$$E = \frac{2p}{4\pi\varepsilon_0\, r^3} \tag{2.8}$$

Eqn. (2.8) shows that electric field is in the same direction of P and its magnitude is inversely proportional to the third power of distance of the observation point from the centre of the dipole.

2.3.2 A Dipole in a uniform Electric Field

A uniform electric field has constant magnitude and fixed direction. Such a field is produced between the plates of a charged parallel plate capacitor. The net force acting on a neutral object placed in a uniform electric field is zero. However, the electric field can produce a net torque if the positive and negative charges are concentrated at different locations on the object. The behavior of an electric dipole can now be examined when it is placed in a uniform electric field (Figure 2.6)

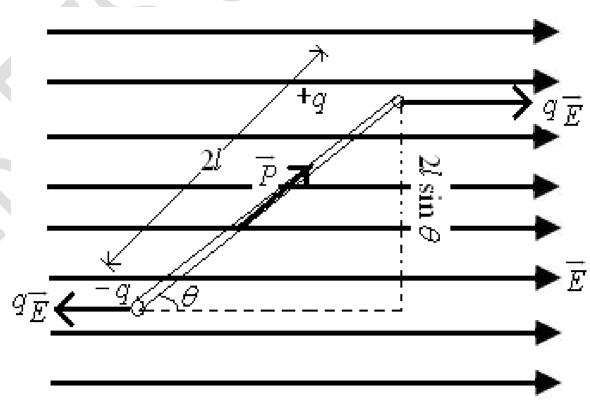


Figure 2.6 : Dipole in a uniform electric field

Let us choose x- axis such that the electric field points along it. Suppose that the dipole axis makes an angle θ with the field direction. A force qE acts on charge +q along the positive x direction and an equal force acts on charge –q in the negative x direction. Two equal, unlike and parallel forces form a couple and tend to rotate the dipole in clockwise direction. This couple tends to align the dipole in the direction of the external electric field E. The magnitude of the torque τ is given by:

$\tau =$ Force x arm of the couple

$$= qE \times 2l \sin\theta \qquad (2.9)$$

In vector form,

$$\tau = p \times E \qquad (2.10)$$

When $\theta = 0$, the torque is zero, and

For $\theta = 90^0$, the torque on the dipole is maximum, equal to pE. It can be concluded that the electric field tends to rotate the dipole and align it along its own direction.

2.3.3 Potential Energy of an Electric Dipole

The work done by the electric field to rotate the dipole by an angle $d\theta$ is

$$dW = -\tau\, d\theta = -p\,E \sin\theta\, d\theta \qquad (2.11)$$

The negative sign indicates that the torque opposes any increase in θ. Therefore, the total amount of work done by the electric field to rotate the dipole from angle θ_0 to θ is:

$$W = \int_{\theta_0}^{\theta} (-p\,E \sin\theta) d\theta = p\,E(\cos\theta - \cos\theta_0) \qquad (2.12)$$

The results shows that a *positive* work is done by the field when $\cos\theta > \cos\theta_0$. The change in potential energy ΔU of the dipole is the negative work done by the field:

$$\Delta U = U - U_0 = -W = -p\,E(\cos\theta - \cos\theta_0) \qquad (2.13)$$

Where $U_0 = -pE\cos\theta_0$ is the potential energy at a reference point. The reference point is chosen to be $\theta_0 = \frac{\pi}{2}$ so that the potential energy is zero $, U_0 = 0$. Thus, in the presence of an external field the electric dipole has a potential energy

$$U = -p\,E\cos\theta = -p.E \qquad (2.14)$$

A system is at a stable equilibrium when its potential energy is a minimum. This occurs when the dipole **p** is aligned parallel to **E** making U to be minimum, $U_{\min} = -p\,E$. On the other hand, when **p** and **E** are anti-parallel, $U_{\max} = +p\,E$ is a maximum and the system is unstable. If the dipole is placed in a non-uniform field, there would be a net force on the dipole in addition to the torque. The resulting motion would be a combination of linear acceleration and rotation.

2.4 Charge Density

The electric field due to a small number of charged particles can be calculated by using the superposition principle. But what happens if we have a very large number of charges distributed in some region of space? Let's consider the system shown in Figure 2.7

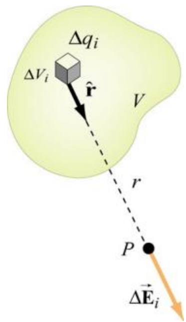


Figure 2.7: Electric field due to a small charge element Δq_i.

2.4.1 Volume Charge Density

Suppose we wish to determine the electric field at some point P. Let's consider a small volume element ΔV_i which contains an amount of charge Δq_i. The distances between charges within the volume element ΔV_i are much smaller than compared to r, the distance between ΔV_i and P. In the limit where ΔV_i becomes infinitesimally small, the volume charge density ρ(**r**) is defined as:

$$\rho(\bar{r}) = \lim_{\Delta v_i \to 0} \frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV} \tag{2.15}$$

The S.I. unit of $\rho(\bar{r})$ (charge/ unit volume) is C/m^3. The total amount of charge within the entire volume V is:

$$Q = \sum_i \Delta q_i = \int_V \rho(\bar{r}) dV \tag{2.16}$$

2.4.2 Surface Charge Density

In a similar manner, the charge can be evenly distributed over a surface S of area A with a *surface charge density* σ which is defined as:

$$\sigma(\bar{r}) = \frac{dq}{dA} \tag{2.17}$$

The S.I. unit of σ (charge/unit area) is C/m^2. The total charge on the entire surface is:

$$Q = \iint_S \sigma(\bar{r}) dA \tag{2.18}$$

2.4.3 Line Charge Density

If the charge is distributed over a length, then the *linear charge density* λ is defined as:

$$\lambda(\bar{r}) = \frac{dq}{dl} \tag{2.19}$$

The S.I. unit of λ (charge/unit length) is C/m. The total charge is an integral over the entire length:

$$Q = \int_{line} \lambda(\bar{r})\,dl \tag{2.20}$$

If charges are uniformly distributed throughout the region, the densities ($\sigma, \rho\, or\ \lambda$) are said to be uniform.

2.4.4 Electric Fields due to Continuous Charge Distributions

The electric field at point P due to continuous charge distribution shown in Figure 2.7 can be evaluated by:

(1) Dividing the charge distribution into small elements containing Δq amount of charge as shown in Figure 2.7

(2) Evaluating the electric field at a point P due to each charge element dq using Coulomb's law:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \tag{2.21}$$

Where r is the distance from dq to P and $\hat{r}$ is the corresponding unit vector.

(3) Evaluating the total electric field at point P due to all such charge elements in charge distribution as follows:

$$E \cong \frac{1}{4\pi\varepsilon_0}\sum_i \frac{\Delta q_i}{r_i^2}\hat{r}_i \tag{2.22}$$

Where index i refers to the i_{th} charge element in the entire charge distribution. Since the charge is distributed continuously over some region, the sum becomes integral. Hence, the total electric field at point P within the limit $\Delta q \to 0$ is:

$$E = \frac{1}{4\pi\varepsilon_0}\lim_{\Delta q_i \to 0}\sum_i \frac{\Delta q_i}{r_i^2}\hat{r}_i = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r} \tag{2.23}$$

And integration is done over the entire charge distribution.

2.4.5 Electric Field Due to a Uniform Line of Charge

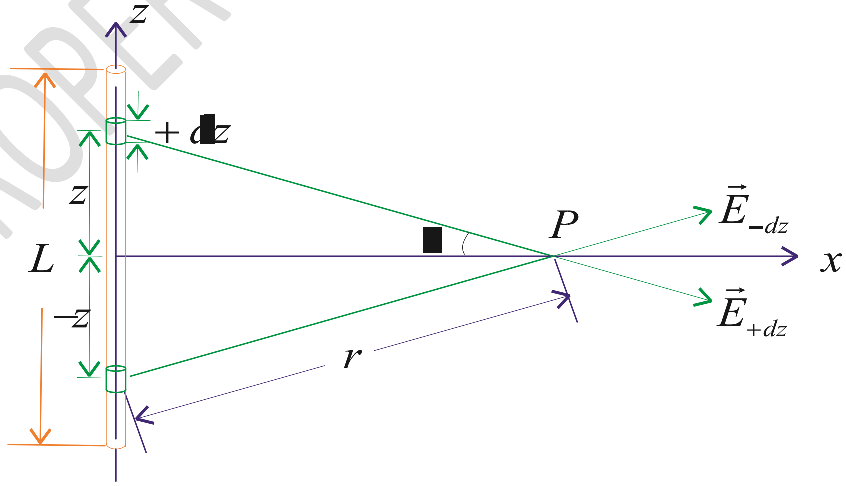


Figure 2.8 Electric field due to a uniform line of charge

The electric field **E** at point P due to a uniform line of charge can be determined from Figure 2.8 By considering the symmetry, the electric field from positive z -axis to negative z – axis cancelled along the z directions. Only the horizontal component of the electric field need to be considered. For each element of length dz charge $dq = \lambda dz$.

$\therefore$ Horizontal E- field at point P due to element $dz = |dE| \cos\theta$

$$|d\vec{E}|\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dz}{r^2}\cos\theta \qquad (2.24)$$

The total electric field due to entire line of charge at point P is:

$$E = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{4\pi\varepsilon_0}.\frac{\lambda\, dz}{r^2}\cos\theta = 2\int_{0}^{\frac{L}{2}} \frac{\lambda}{4\pi\varepsilon_0}.\frac{dz}{r^2}\cos\theta \qquad (2.25)$$

To calculate this integral, use:

Change of variable (from z to θ) and noting that x is fixed, but z, r, θ all varies.

$$z = x\tan\theta \quad \therefore \; dz = x\sec^2\theta\, d\theta \qquad (2.26a)$$

$$x = r\cos\theta \quad r^2 = x^2\sec^2\theta \qquad (2.26b)$$

When $z = 0 \quad \theta = 0^0$; $z = \frac{L}{2} \quad \theta = \theta_0$ where $\tan\theta_0 = \frac{\frac{L}{2}}{x}$ (2.26c)

Equation (2.25) then becomes:

$$E = 2.\frac{\lambda}{4\pi\varepsilon_0}\int_0^{\theta_0}\frac{x\sec^2\theta\, d\theta}{x^2\sec^2\theta}.\cos\theta$$

$$= 2.\frac{\lambda}{4\pi\varepsilon_0}\int_0^{\theta_0}\frac{1}{x}\cos\theta\, d\theta$$

$$= 2.\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{x}(\sin\theta_0)$$

$$= \frac{1}{4\pi\varepsilon_0}.\frac{\lambda L}{x\sqrt{x^2 + \left(\frac{L}{2}\right)^2}} \quad \text{along x-direction} \qquad (2.27)$$

Using the fact that the total charge $Q = \lambda L$, equation (2.27) becomes:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{x\sqrt{x^2 + \left(\frac{L}{2}\right)^2}} \tag{2.28}$$

When P is a very far point from the rod, x >> L, the term $\left(\frac{L}{2}\right)^2$ can be neglected in the denominator of equation (2.28). Thus, we obtain:

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{x^2} \tag{2.29}$$

Therefore, the system behaves like a point charge.

If L >> x, the term x^2 can be neglected in the denominator and we obtain the expression for the electric field due to infinitely long line of charge as:

$$E = \frac{\lambda}{2\pi\varepsilon_0 x} \tag{2.30}$$

2.4.6 Electric Field on the Axis of a Ring

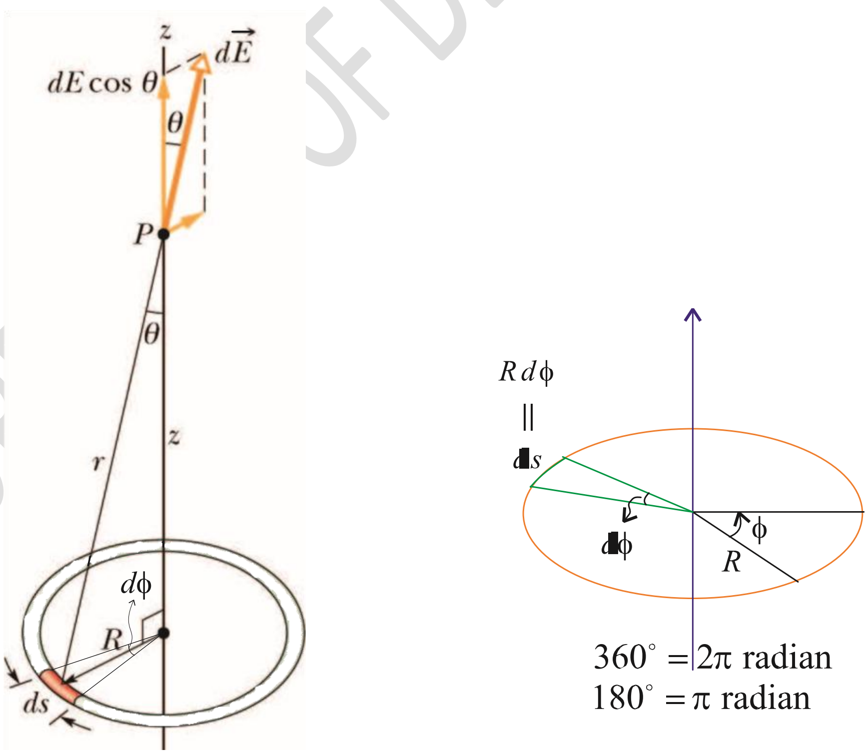


Figure 2.9 Electric Field at a height z above a ring of charge of radius R

Figure 2.9 shows a ring with radius R. From symmetry, for every charge element *dq* considered, there exists dq' where the horizontal electric field components cancel. Therefore, the overall electric field lies along z- direction. For each element of length *dz*, the charge *dq* is:

$$dq = \lambda . ds \tag{2.31}$$

Where λ = linear charge density and ds = circular length element

But $ds = R\, d\phi$ (2.32)

Where ϕ is the angle measured on the ring plane.

The net electric field along z-axis due to *dq* is:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta \tag{2.33}$$

The total electric field is:

$$E = \int dE$$

$$= \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} . \frac{\lambda R\, d\phi}{r^2}\cos\theta \qquad \left(\cos\theta = \frac{z}{r}\right) \tag{2.34}$$

In this case, θ,*R* and *r* are *fixed* as ϕ varies, therefore, r, θ is converted to R, Z. Equation (2.34) now becomes:

$$E = \frac{1}{4\pi\varepsilon_0} . \frac{\lambda R Z}{r^3}\int_0^{2\pi} d\phi \tag{2.35}$$

$$E = \frac{1}{4\pi\varepsilon_0} . \frac{\lambda(2\pi R) Z}{\left(Z^2 + R^2\right)^{\frac{3}{2}}} \tag{2.36}$$

But the total charge on the ring is: $Q = \lambda(2\pi R)$ (2.37)

$$E = \frac{1}{4\pi\varepsilon_0}\frac{QZ}{\left(Z^2 + R^2\right)^{\frac{3}{2}}} \tag{2.38}$$

2.4.7 Electric Field Due to a Uniformly Charged Disk

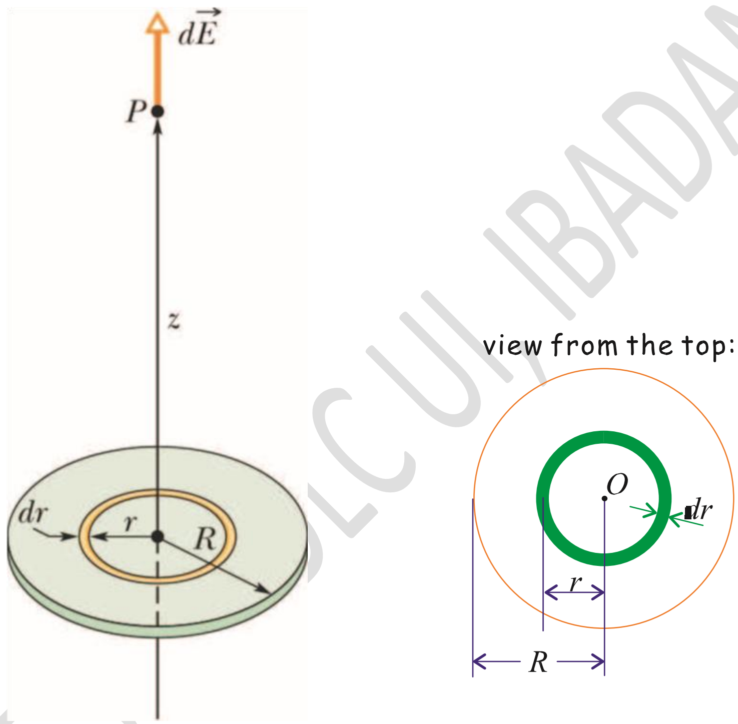


Figure 2.10 A uniformly charged disk of radius R

Figure 2.10 shows a disk of radius R which has a uniform positive surface charge density σ. It is assumed that a point P lies at a distance from the disk along its central perpendicular axis. The electric field at P can be obtained by dividing the disk into concentric rings. Then the electric field at P for each ring is calculated and finally we sum up the contributions of all the rings. Figure 2.10 shows one of such ring with radius r, radial width dr and surface area A. $A = 2\pi r\,dr$. Since σ is the charge per unit area, then the charge *dq* on this ring is:

$$dq = \sigma\, dA = \sigma(2\pi r\, dr) \qquad (2.39)$$

The electric field of a disk is then calculated by integrating concentric rings of charges.

The electric field from ring, $dE = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{dq\, z}{\left(z^2 + r^2\right)^{\frac{3}{2}}}$ (2.40)

$$\therefore\ E = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{2\pi\sigma r\,dr\,z}{\left(z^2+r^2\right)^{\frac{3}{2}}} = \frac{1}{4\pi\varepsilon_0}\int_0^R 2\pi\sigma z\,\frac{r\,dr}{\left(z^2+r^2\right)^{\frac{3}{2}}} \tag{2.41}$$

Introducing change of variable and change of integration limit gives:

$$u = z^2 + r^2 \quad \Rightarrow \left(z^2+r^2\right)^{\frac{3}{2}} = u^{\frac{3}{2}} \tag{2.42a}$$

$$d\,u = 2r\,dr \quad \Rightarrow r\,dr = \frac{1}{2}\,du \tag{2.42b}$$

Also, when $r = 0 \quad , u = z^2$

$$r = R \quad , u = z^2 + R^2 \tag{2.42c}$$

$$\therefore E = \frac{1}{4\pi\varepsilon_0}\,.\,2\pi\sigma z \int_{z^2}^{z^2+R^2} \frac{1}{2} u^{\frac{-3}{2}}\,du \tag{2.43}$$

But $\int u^{-\frac{3}{2}}\,du = -2u^{\frac{-1}{2}}$

$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \tag{2.44}$$

If R >> Z , that is, if we have an infinite sheet of charge with charge density σ :

$$E \approx \frac{\sigma}{2\varepsilon_0} \tag{2.45}$$

2.5 Motion of Charged Particle in a Uniform Electric Field

When a particle of charge q and mass m is in an external electric field $\overline{E}$, a force $q\overline{E}$ will be exerted on this particle. If the magnitude of the only force acting on this particle is $q\overline{E}$, then according to Newton's second law, the acceleration of the particle will be given by:

$$\vec{a} = \frac{q\vec{E}}{m} \tag{2.46}$$

If $\vec{E}$ is uniform, then $\vec{a}$ will be a constant vector.

Consider a charge $+q$ moving between two parallel plates of opposite charges, as shown in Figure 2.11

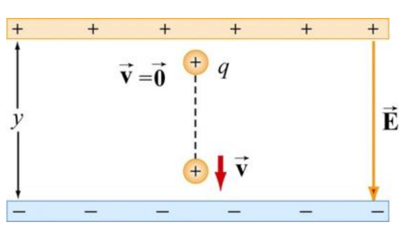


Figure 2.11 Charge moving in a constant electric field

Let the electric field between the plates be $\mathbf{E} = -E_y \hat{\mathbf{j}}$, with $E_y > 0$. The charge will experience a downward Coulomb force given by:

$$\mathbf{F}_e = q\mathbf{E} \tag{2.47}$$

Suppose the particle is at rest ($v_0 = 0$) when it is first released from the positive plate.

The final speed v of the particle as it strikes the negative plate is:

$$v = \sqrt{\frac{2qEy}{m}} \tag{2.48}$$

where y is the distance between the two plates. The kinetic energy of the particle when it strikes the plate is:

$$K = \frac{1}{2}mv_y^2 = qE_y y \tag{2.49}$$

Worked Examples

2.1 An object with a net charge of 24μC is placed in a uniform electric field of 610 N/C directed vertically upward. Calculate the mass of this object if it "floats" in the field.

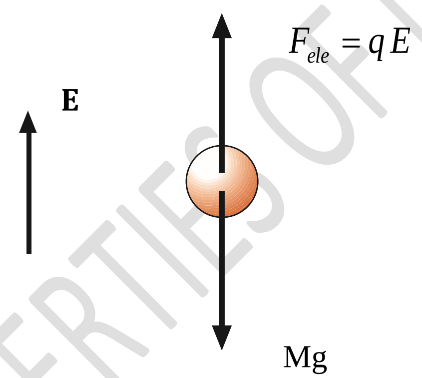


Figure 2.12

The forces acting on the mass are shown in figure 2.12 The weight point downward and has magnitude mg(m is the mass of the object) and the electrical force acting on the mass has magnitude $F = |q|\,E$ where q is the charge of the object and E is the magnitude of the electric field. The object "floats", so the net force is zero. This gives:

$$|q|\,E = mg \quad \Rightarrow m = \frac{|q|\,E}{g} = \frac{(24\times10^{-6}\,C)(610\,N\,C^{-1})}{(9.80\,m\,s^{-2})}$$
$$= 1.5\,g$$

The mass of the object is 1.5 g

2.2 A point charge creates an electric field of 1.00 N/C at a point 1.0 m away. Calculate the magnitude of the point charge.

Solution:

The magnitude of the electric field E due to a point charge *q* at a distance *r* is given by:

$$E = k\frac{|q|}{r^2}$$

$$|q| = \frac{E\,r^2}{k} = \frac{(1.00\ N\,C^{-1})(1.0\,m)^2}{(9.00\times 10^9\ N.m^2\,C^{-2})}$$

$$= 1.11\times 10^{-10}\ C$$

The magnitude of the charge is $1.11\times 10^{-10}\ C$

2.3 A disk of radius 2.5 cm has a surface charge density of $5.3\,\mu C\,m^{-2}$ on its upper face. Calculate the magnitude of the electric field produced by the disk at a point on its central axis at distance *z* = 12cm from the disk.

Solution:

Given that: $r = 2.5\times 10^{-2}$ m, $\sigma = 5.3\,\mu C\,m^{-2}$ and $z = 12\times 10^{-2}$ m

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right)$$

$$= \frac{5.3\times 10^{-6}\ \mu C\,m^{-2}}{2(8.85\times 10^{-12}\ N^{-1}\ m^{-2}\ C^2)}\left(1 - \frac{(12\times 10^{-2}\ m)}{\sqrt{(12\times 10^{-2}\ m)^2 + (2.5\times 10^{-2}\ m)^2}}\right)$$

$$= 629\ N\,C^{-1}$$

At the given point, the electric field has magnitude 629 N/C and points away from the disk.

2.4 An electron and a proton are each placed in an electric field of 520 N/C. Calculate the speed of each particle 48 ns after being released.

Solution:

Consider the electron. From F = *q***E**, and the fact that the magnitude of the electron's charge is 1.60×10^{-19} C, the magnitude of the force on the electron is:

$$F = |q|\,E = (1.60\times 10^{-19}\ C)(520\ N\,C^{-1}) = 8.32\times 10^{-17}\ N$$

The mass of electron , $m_e = 9.11\times 10^{-31}\ kg$, using Newton's second law:

The magnitude of acceleration $a = \frac{F}{m} = \frac{8.32\times 10^{-17}\ N}{9.11\times 10^{-31}\ kg} = 9.13\times 10^{13}\ ms^{-2}$

Since the electron starts from rest, $v_0 = 0$ and the final speed after 48 ns is:

$$v = at = (9.13\times 10^{13}\ ms^{-2})(48\times 10^{-9}\ s) = 4.4\times 10^{6}\ ms^{-1}$$

The magnitude of the proton's charge is the same as that of the electron, the magnitude of the force is the same.

The magnitude of its acceleration $a = \frac{F}{m_p} = \frac{(8.32 \times 10^{-17}\ N)}{(1.67 \times 10^{-27}\ kg)} = 4.98 \times 10^{10}\ ms^{-2}$

The final speed after 48 ns is:

$v = at\ = (4.98 \times 10^{10}\ ms^{-2})(48 \times 10^{-9}\ s) = 2.4 \times 10^{3}\ ms^{-1}$

The proton's final speed is $2.4 \times 10^{3}\ ms^{-1}$.

2.5 Calculate the magnitude of the torque acting on an electric dipole aligned at angle 30^{0} with the direction of a uniform electric field of magnitude $5 \times 10^{4}\ NC^{-1}$.The dipole moment of the electric dipole is $4 \times 10^{-9}\ Cm$.

Solution:

Electric dipole moment, $p = 4 \times 10^{-9}\ Cm$

Electric field $E = 5 \times 10^{4}\ NC^{-1}$

Torque acting on the dipole,

$\tau = |p||E|\sin\theta = (4 \times 10^{-9}\ Cm)(5 \times 10^{4}\ NC^{-1})\sin 30^{0} = 10^{-4}\ Nm$

Therefore, the magnitude of the torque acting on the dipole is $10^{-4}\ Nm$

2.6 Two charges $q_A = 2.5 \times 10^{-7}\ C$ and $q_B = -2.5 \times 10^{-7}\ C$ are placed at the position: A (0, 0, -15 cm) and B (0, 0, + 15 cm), respectively. Calculate the dipole moment of the system.

Solution:

The charges which are located at the given points are shown in the co-ordinate system as:

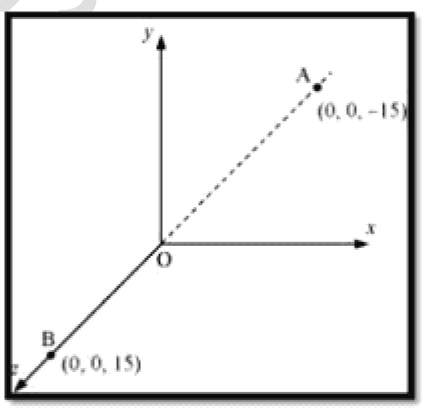


Figure 2.13

The charge at point A, $q_A = 2.5 \times 10^{-7}\ C$

The charge at point B, $q_B = -2.5 \times 10^{-7}\ C$

The distance between the two charges at point A and B is: $d = 15 + 15 = 30\,cm$

Electric dipole moment of the system is given by:

$$p = q_A \times d = q_B \times d = \left(2.5 \times 10^{-7}\ C\right)\left(0.3\ m\right) = 7.5 \times 10^{-8}\ C\,m$$

Therefore, the electric dipole moment of the system is $7.5 \times 10^{-8}\ C\,m$ along positive z – axis.

Self-Assessment Questions (SAQs)

1. An electron is released from rest in a uniform electric field of magnitude $2.0 \times 10^{4}\ N\,C^{-1}$. Calculate the acceleration of the electron. (Ignore gravitation).

2. Calculate the magnitude and direction of the electric field at the centre of the square as shown in the figure below, if $q = 1.0 \times 10^{-8}$ C and $a = 5.0$cm

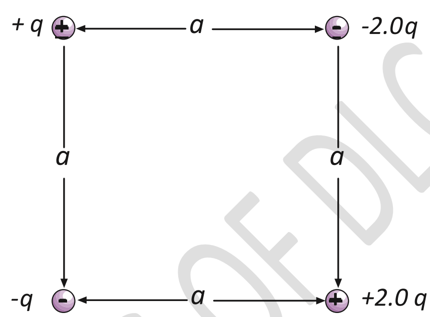


Fig. 2.14: Charge configuration

3. An electron is constrained to the central axis of the ring of charge with radius R and total charge q. Show that the electrostatic force exerted on the electron can cause it to oscillate through the centre of the ring with an angular frequency

$$\omega = \sqrt{\frac{e\,q}{4\pi\,\varepsilon_0\,m\,R^3}}$$

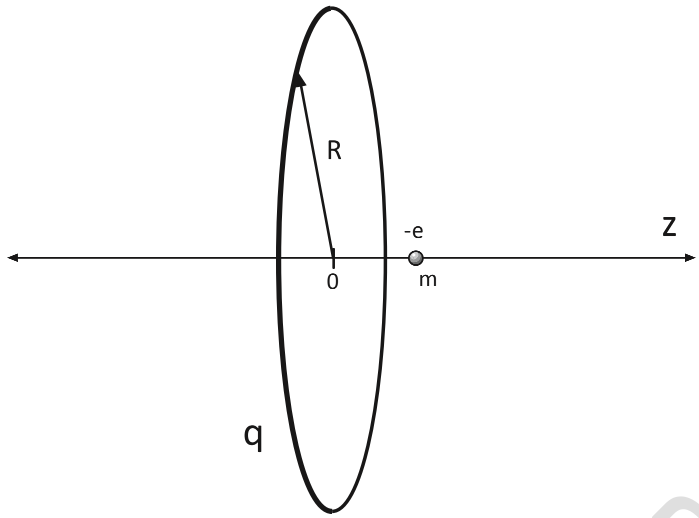


Fig. 2.15: Electron oscillates on z axis through centre of charged ring of radius R and total charge q.

4. The electrons in a particle beam each has a kinetic energy of 1.60×10^{-17} J. Calculate the magnitude and direction of the electric field that will stop these electrons in a distance of 10.0 cm.

5. Two massless point charges +9Q and -Q are fixed on the x-axis at x = -d and x = d:

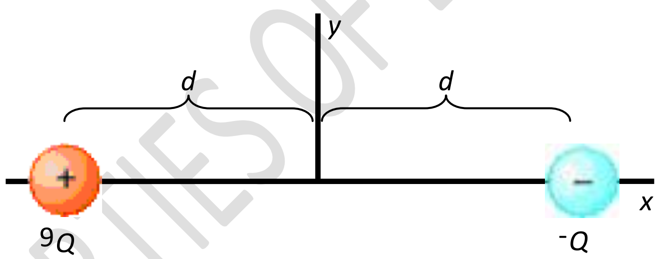


Figure 2.16

There is one point on the x-axis, x = x_0, where the electric field is zero. What is x_0?

6. Two point charges, $q_A = 3\ \mu C$ and $q_B = -3\ \mu C$ are separated by a distance 20 cm when placed in a vacuum. Calculate the magnitude of the electric field at a point O of the line joining the two charges.

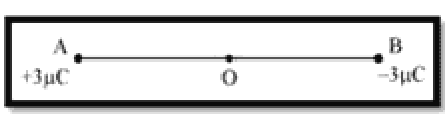


Figure 2.17

7. A conducting sphere of radius 10 cm has an unknown charge. If the electric field 20 cm from the centre of the sphere is $1.5 \times 10^3 \ NC^{-1}$ and points radially inward, what is the net charge on the sphere?

8. An infinite line charge produces a field of $9 \times 10^4 \ NC^{-1}$ at a distance of 2 cm. Calculate the linear charge density.

SUMMARY

In this session, you have learnt that:

1. The electric field due to a charge *q* at a point in space is defined as the force experienced by a unit test charge q_0:

$$\vec{E} = \frac{\vec{F}}{q_0}$$

2. The electric field at a distance *r* from a charge *q* is :

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

3. Using the superposition, the total electric field due to a group of charges is equal to the vector sum of the electric fields of individual charges:

.

$$\vec{E} = \sum_i \vec{E_i} = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \hat{r}$$

4. An electric dipole is a system of two equal and opposite charges separated by a small distance. The electric dipole moment vector P points from the negative charge to the positive charge and is given by:

$$P = q \times 2a$$

5. Electric field line is an imaginary line drawn in such a way that the direction of its tangent at any point is the same as the direction of the electric field vector.
6. The torque acting on an electric dipole placed in a uniform electric field **E** is given by:

$$\tau = p \times E$$

7. The potential energy of an electric dipole in a uniform external electric field **E** is given by:

$$U = -pE\cos\theta = -p.E$$

8. The electric field at a point in space due to a continuous charge element dq is given by:

$$E = \frac{1}{4\pi\varepsilon_0} \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^2}\hat{r}_i = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}$$

9. A particle of mass *m* and charge *q* moving in an electric field **E** has an acceleration *a* given by:

$$\vec{a} = \frac{q\vec{E}}{m}$$

Study Session 3: Electric Flux and Gauss's Law

Expected Duration: 1 week or 2 contact hours

Introduction

In the last session, you learnt that Coulomb's law is used to determine the electric field of a continuous charge distribution. However, the electric field of a continuous charge distribution can become very complicated to determine for some charge distribution. Gauss's law is introduced as an alternative method for calculating electric fields of certain highly symmetrical charge distribution. In this session, you will learn about Gauss's law and its application. The concept of electric flux are discussed.

Learning Outcomes

When you have studied this session, you should be able to explain the:

3.1 Electric Flux

3.2 Gauss's Law

3.3 Electric Field calculation with Gauss's Law

3.1 Electric Flux

Flux is a qualitative measure of the number of lines of a vector field that passes perpendicularly through a surface. Electric flux $\Phi\ E$ represents the number of electric field lines crossing a surface. Figure 3.1 shows an electric field **E** passing through a portion of a surface of area A. The area of the surface is represented by a vector **A**, whose magnitude is the area A of the surface, and whose direction is perpendicular to the surface. The electric flux is defined as:

$$\phi_E = E.A = E\,A\cos\theta \tag{3.1}$$

It is a quantitative measure of the number of lines of **E** that pass normally through the surface area **A.** The number of lines represents the strength of the field

Mathematically:

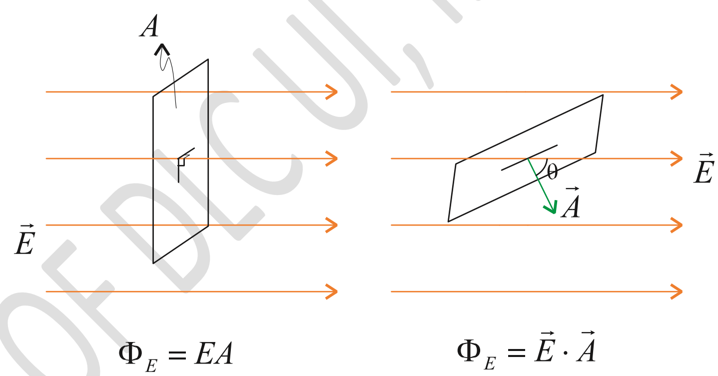


Figure 3.1 Electric flux

3.2 Gauss's Law

It was discussed in section 3.1 that an electric flux was a quantitative measure of the number of electric field lines passing normally through an area. Let us now consider the amount of electric flux that emanates from a positive point charge. Figure 3.2 shows a positive point charge surrounded by an imaginary spherical surface called a Gaussian surface. It is difficult to measure the amount of electric flux through the sphere using equation 3.1 because the direction of the electric field is different at every point. Instead, the spherical surface is broken up into a large number of infinitesimal surface areas dA, as shown in Figure 3.2b and the infinitesimal amount of flux $d\phi_E$ through each of these small areas is calculated. That is:

$$d\phi_E = E.dA \tag{3.2}$$

The total flux out of the Gaussian surface is the sum or integral of all the infinitesimal fluxes $d\phi_E$, through all the infinitesimal areas dA, that is,

$$\phi_E = \oint d\phi_E = \oint E.dA \tag{3.3}$$

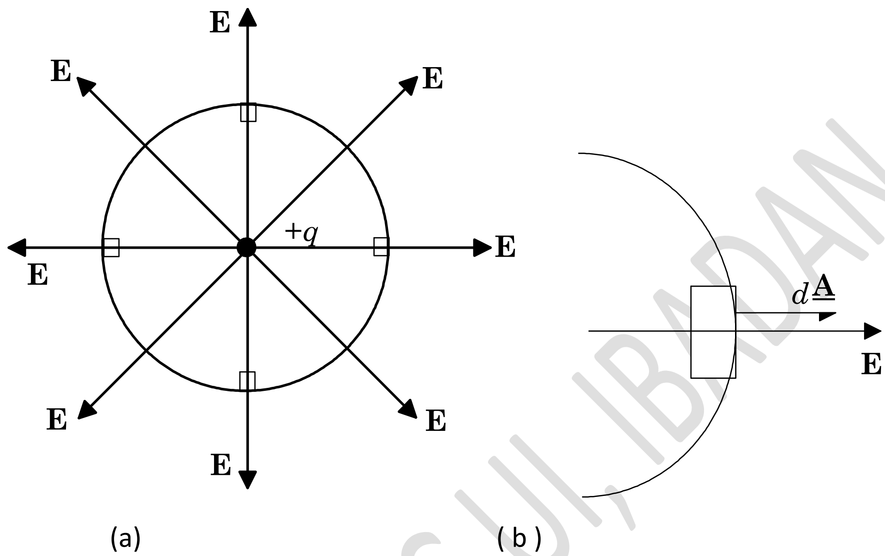


Figure 3.2 Gaussian Surface

The integral symbol $\oint$ means that the integration is performed over the entire closed surface that the flux is passing through. The electric field vector **E** is everywhere radial from the point charge q, and dA is also everywhere radial, hence

$$\phi_E = \oint E.dA = \oint E.dA\cos 0^0 \tag{3.4}$$

Thus,

$$\phi_E = \oint E\,dA \tag{3.5}$$

It was discussed in session 2, that the electric field of a point charge was given by:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = k\frac{q}{r^2}\hat{r} \tag{2.2}$$

Substituting the electric field of a point charge from eqn. (2.2) into eqn. (3.5) gives the electric flux through the spherical surface.

$$\phi_E = \oint k\frac{q}{r^2}.dA \tag{3.6}$$

Where r is the radius of the spherical Gaussian surface which is constant for the sphere. Therefore,

$$\phi_E = k\frac{q}{r^2}\oint dA \tag{3.7}$$

But the integral of all the elements of area dA is equal to the entire surface area of the sphere. Since the area of a sphere is $4\pi r^2$, therefore,

$$\oint dA = 4\pi r^2 \tag{3.8}$$

Thus, the electric flux emanating from a point charge becomes:

$$\phi_E = k\frac{q}{r^2}\left(4\pi r^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(4\pi r^2\right) \tag{3.9}$$

Hence, the electric flux associated with a point charge is:

$$\phi_E = \frac{q}{\varepsilon_0} \tag{3.10}$$

Equation (3.10) is Gauss's law. It states that the net electric flux through a closed Gaussian surface is equal to the total charge q inside the surface divided by ε_0.

Gauss's law is an alternative method used to calculate the electric field of charge distribution.

3.3 Electric Field Calculation with Gauss's Law

Gauss's law can be used to calculate the electric field of any spherically symmetric distribution of charge. A spherically symmetric distribution of charge means that the number of charges per unit volume depends only on the radius from a central point. The Gauss's law is applied to calculate the electric field of the following distribution of charges.

3.3.1 Electric Field due to an Infinite Line of Charge

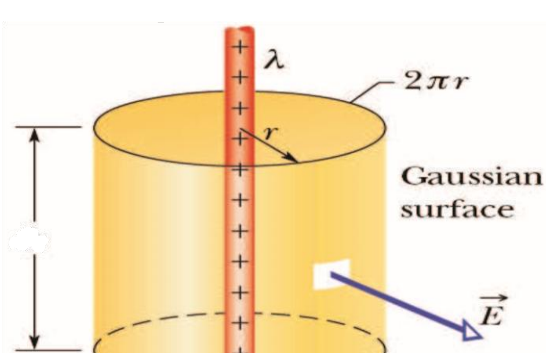


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Figure 3.3 Electric Field due to an Infinite Line of Charge having uniform linear charge density.

A line charge is in the form of a thin charged wire of infinite length with a uniform charge density λ (charge per unit length). The electric field intensity **E** at a distance *r* from an infinite line of charge shown in figure 3.3 can be determined using Gauss's law.

To determine the electric field, we must first draw a Gaussian surface of radius *r* and length L around the infinite line of charge as shown in figure 3.3.The surface of this cylinder is the Gaussian surface. The magnitude of the electric field intensity **E** is the same at every point on the curved surface of the cylinder because all points are at the same distance from the charged wire. The electric field direction and the normal to area element dA are parallel. Let the length of the Gaussian cylinder be L. The total charge enclosed in the cylinder is $q = \lambda L$. The area of the curved surface of the cylinder is $2\pi r L$. The flat surfaces at the top and bottom of the cylinder do not contribute to the total flux. Hence,

$$\phi_E = E.A = E \times 2\pi r L \tag{3.11}$$

According to Gauss's law: $\phi_E = \dfrac{q}{\varepsilon_0}$

$$E \times 2\pi r L = \frac{q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \tag{3.12}$$

$$E = \frac{\lambda}{2\pi \varepsilon_0 r} \tag{3.13}$$

Equation (3.13) shows that electric field varies inversely with distance.

3.3.2 Electric Field due to a Plane Sheet of Charge

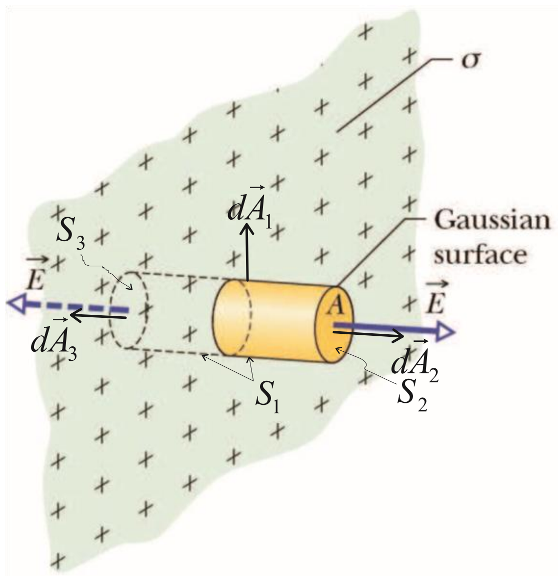


Figure 3.4 Electric Field due to a Plane Sheet of Charge

Using Gauss's law, the electric field intensity **E** at a distance *r* of an infinite plane sheet of charge shown in figure 3.4 can be determined. Consider a thin infinite plane sheet of charge having surface charge density σ (charge per unit area). It is assumed that the charge is uniformly distributed over the sheet. To determine the electric field, we draw a cylindrical Gaussian surface **S** through the sheet of charge. By symmetry, the electric field **E** on either side of sheet is perpendicular to the plane of the sheet and have the same magnitude at all points equidistant from the sheet. Gauss's law for the total flux emerging from the Gaussian cylinder is:

$$\phi_E = \oint_S E.dA = \frac{A\sigma}{\varepsilon_0} \tag{3.14}$$

$$\oint_{S_1} E.dA = 0 \text{ (E is perpendicular to } dA \text{ over the whole surface S}_1\text{)} \tag{3.15}$$

$$\oint_{S_2} E.dA + \oint_{S_3} E.dA = 2EA \tag{3.16}$$

E and dA are parallel to each other and the angle between E and dA is zero.

$$\therefore\ 2EA = \frac{A\sigma}{\varepsilon_0} \tag{3.17}$$

$$E = \frac{\sigma}{2\varepsilon_0} \tag{3.18}$$

Equation (3.18) shows that the magnitude of electric field depend only on the surface charge density and not on the distance from the sheet of charge.

Worked Examples

3.1 A 4.0 cm –square in the x-y plane sits in a uniform electric flux $E = (2.0i + 3.0j + 5.0k)NC^{-1}$. Calculate the electric flux through the square.

Solution:

Since the square is in the x-y plane, only electric field in the perpendicular z- direction contributes to the flux. Therefore:

$$\phi_E = E \perp A = (5.0\,NC^{-1}) \times (16 \times 10^{-4}\ m^2) = 8.0 \times 10^{-3}\ N.m^2\,C^{-1}$$

3.2 A point charge of $2.0\,\mu C$ is at the centre of a cubic Gaussian surface 9.0 cm on edge. Determine the net electric flux through the surface.

Solution:
Net electric flux, ϕ_E through the cubic surface is given by:

$$\phi_E = \frac{q}{\varepsilon_0} = \frac{(2.0 \times 10^{-6}\,C)}{(8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2})} = 2.26 \times 10^5\ N.m^2\,C^{-1}$$

The net electric flux through the surface of the cube is $2.26 \times 10^5\ N.m^2\,C^{-1}$

3.3 A 6.0 nC point charge is located at the centre of a cube of side length 2.0 m. Calculate the electric flux through each of the faces of the cube.

Solution:

By Gauss's law, the total flux coming out of the cube is:

$$\phi_E = \frac{q}{\varepsilon_0} = \frac{(6.0 \times 10^{-9}\ C)}{(8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2})} = 678\ N.m^2\ C^{-1}$$

By symmetry, this flux must be evenly split between the six faces. Therefore, the flux through each face is $\frac{1}{6}(678\ N.m^2\ C^{-1}) = 114\ N.m^2\,C^{-1}$

3.4 A long copper wire with radius of 1.0 mm carries a uniform surface charge density of 5.0 $\mu C\,m^{-2}$. Find the:

(a) total charge in a 1.0 metre- long section of the wire.

(b) magnitude of the electric field at a distance of 15 cm from the wire.

Solution:

(a) A 1.0 m long section of wire has a surface area of:

$$A = 2\pi r h = 2\pi\left(1.0\times10^{-3}\ m\right)\left(1.0\,m\right) = 6.28\times10^{-3}\ m^2$$

And therefore has a charge of $q = \sigma A = \left(5.0\times10^{-6}\ C\,m^{-2}\right)\left(6.28\times10^{-3}\ m^2\right)$

$$= 3.14\times10^{-8}\ C$$

(b) The linear charge density of the wire is $\lambda = 3.14\times10^{-8}\ C\,m^{-1}$. Therefore, the electric field at a distance of 15 cm is:

$$E = \frac{\lambda}{2\pi\,\varepsilon_0\,r} = \frac{\left(3.14\times10^{-8}\ C\,m^{-1}\right)}{2\,\pi\left(8.85\times10^{-12}\ C^2\ N^{-1}\ m^{-2}\right)} = 3.8\times10^{3}\ N\,C^{-1}$$

Self- Assessment Questions

1. Consider a uniform electric field $E = \left(3.0\times10^{3}\ i\right)N\,C^{-1}$. Determine the:

   (a) flux of this field through a square of side 10 cm on a side whose plane is parallel to the y-z plane.

   (b) flux through the same square if the normal to its plane makes an angle 60^0 with the x-axis.

2. A point charge causes an electric flux of $-1.0\times10^{3}\ N.m^2\,C^{-1}$ to pass through a spherical Gaussian surface of 10.0 cm radius centered on the charge.
   (a) If the radius of the Gaussian surface were doubled, how much flux would pass through the surface?
   (b) What is the value of the point charge?
3. A conducting sphere of radius 10 cm has an unknown charge. If the electric field 20 cm from the centre of the sphere is $1.5\times10^{3}\ N\,C^{-1}$ and points radially inward, what is the net charge on the sphere?
4. A uniformly charged conducting sphere of 2.4 m diameter has a surface charge density of $80.0\ \mu C\,m^{-2}$. Calculate the magnitude of the:
   (a) Charge on the sphere
   (b) Total electric flux leaving the surface of the sphere

SUMMARY

In this session, you have learnt that:

1. Electric flux ΦE represents the number of electric field lines crossing a surface and is defined as $\phi_E = E.A = E\,A\cos\theta$.
2. Gauss's law states that the net electric flux through a closed Gaussian surface is equal to the total charge q inside the surface divided by ε_0. That is:

$$\phi_E = \frac{q}{\varepsilon_0}$$

Gauss's law is an alternative method used to calculate the electric field of charge distribution.

3. The electric field due to an infinite line of charge is given by:

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

4. The electric field due to a plane sheet of charge is given by:

$$E = \frac{\sigma}{2\varepsilon_0}$$

Study Session 4: Electric Potential and Electrical Potential Energy

Expected Duration: 1 week or 2 contact hours

Introduction

Electric potential is a location-dependent quantity that expresses the amount of potential energy per unit charge at a point. In this session, you will learn about the concept of electric potential and electrical potential energy as applied in the study of electrostatics .Equipotential surfaces will also be discussed.

Learning Outcomes

When you have studied this session, you should be able to explain the:

4.1 Electrical Potential energy and conservative forces

4.2 Electric Potential

4.3 Equipotential surfaces

4.4 Relation between Electric field and Electric potential

4.1 Electrical Potential Energy and Conservative Forces

In the introductory mechanics course, a conservative force was explained. A force is said to be conservative if the work done on a particle by the force is independent of the path taken. Electric force is an example of conservative forces. The work done by the electric force as charge moves through an

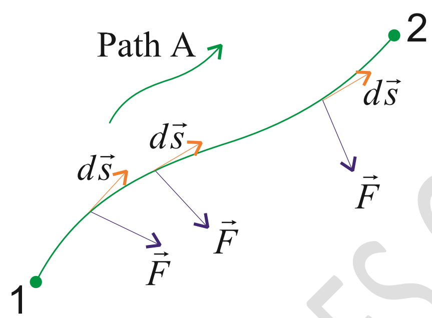


Figure 4.1 a

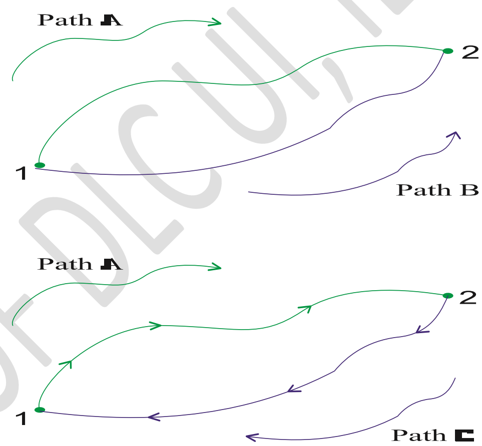


Figure 4.1 b

Infinitesimal distance $d\vec{s}$ along path A as shown in Figure 4.1a is dW.

$$dW = \vec{F}.d\vec{s} \tag{4.1}$$

Therefore, the total work done W by a force $\vec{F}$ in moving the particle from point 1 to point 2 as shown in Figure 4.1b is:

$$W = \int_1^2 \vec{F}.d\vec{s} \text{ (For path A)} \tag{4.2}$$

$d\vec{s}$ is in the tangent direction to the curve of path A.

For conservative forces,

$$\int_1^2 \vec{F}.d\vec{s} \text{(For path A)} = \int_1^2 \vec{F}.d\vec{s} \quad \text{(For path B)} \tag{4.3}$$

Let's consider a path starting at point 1 to 2 through path A and from 2 to 1 through path C.

$$W = \int_1^2 \vec{F}.d\vec{s} \text{ (for path A)} + \int_2^1 \vec{F}.d\vec{s} \text{ (for path C)}$$

$$= \int_1^2 \vec{F}.d\vec{s} \text{ (for path A)} - \int_1^2 \vec{F}.d\vec{s} \text{ (for path B)} \qquad (4.4)$$

$$= 0$$

Equation (4.4) shows that the work done by a conservative force on a particle when it moves around a close path returning to its initial position is zero. This is the basis for defining a conservative force. Since the work done by a conservative force $\vec{F}$ is path independent, then $\nabla \times F = 0$. We can define a quantity, potential energy that depends only on the position of the particle. We define potential energy U such that:

$$dU = -W = -\int \vec{F}.d\vec{s} \qquad (4.5)$$

Therefore, for particle moving from position 1 to position 2:

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}.d\vec{s} = -W \qquad (4.6)$$

Where U_1 and U_2 are potential energy at positions 1 and 2.

Suppose charge q_2 moves from position 1 to position 2 as shown in Figure 4.2 below, the change in potential energy dU can be derived as follows:

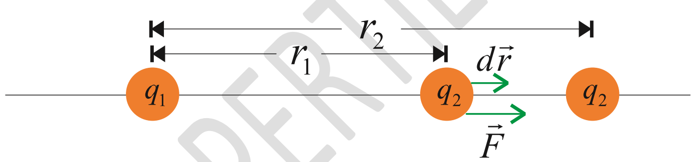


Figure 4.2

From the definition: $U_2 - U_1 = -\int_1^2 \vec{F}.d\vec{r}$

$$= -\int_{r_1}^{r_2} F\, dr \quad \left(\vec{F} \parallel d\vec{r}\right)$$

$$= -\int_{r_1}^{r_2} \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}\, dr$$

$$\Delta U = -\Delta W = \frac{1}{4\pi\varepsilon_0} q_1\, q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{4.7}$$

The following points should be noted:

1. If q_2 moves away from q_1, then $r_2 \;\phi\; r_1$, we have:
   - If q_1 and q_2 are of the same sign, then $\Delta U \;\pi\; 0,\; \Delta W \;\phi\; 0$
     ($\Delta W =$ work done by electric repulsive force)
   - If q_1 and q_2 are of different sign, then $\Delta U \;\phi\; 0, \Delta W \;\pi\; 0$
     ($\Delta W =$ work done by electric attractive force)

2. If q_2 moves towards q_1, then $r_2 \;\pi\; r_1$, we have:
   - If q_1 and q_2 are of the same sign, then $\Delta U \;\phi\; 0, \Delta W \;\pi\; 0$
     ($\Delta W =$ work done by electric repulsive force)
   - If q_1 and q_2 are of different sign, then $\Delta U \;\pi\; 0,\; \Delta W \;\phi\; 0$
     ($\Delta W =$ work done by electric attractive force)
3. The difference in potential energy is important.
   Reference point: $U(r=\infty)=0$

$$\therefore\; U_\infty - U_1 = \frac{1}{4\pi\varepsilon_0} q_1\, q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{1}{4\pi\varepsilon_0} q_1\, q_2 \left(\frac{1}{\infty} - \frac{1}{r_1}\right)$$

$$U(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1\, q_2}{r} \tag{4.8}$$

   If q_1 and q_2 are of the same sign, then, $U(r) \,\phi\, 0$ for all r.
   If q_1 and q_2 are of different sign, then, $U(r) \,\pi\, 0$ for all r.
4. Conservation of Mechanical Energy:
   For a system of charges with no external force,
   E = K + U = constant
   Or
   $\Delta E = \Delta K + \Delta U = 0$ (K = kinetic energy; U = potential energy) (4.9)
   We can obtain an expression for the potential energy of system consisting of three charges q_1, q_2, q_3 as follows:
   - Assume that charges q_1, q_2, q_3 all at $r=\infty$ will have $U=0$
   - Move q_1 from ∞ to its position $\Rightarrow U = 0$
   - Move q_2 from ∞ to its new position $\Rightarrow U = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_1\, q_2}{r_{12}}$
   - Move q_3 from ∞ to its new position $\Rightarrow$ Total P.E

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right] = U_{12} + U_{13} + U_{23} \tag{4.10}$$

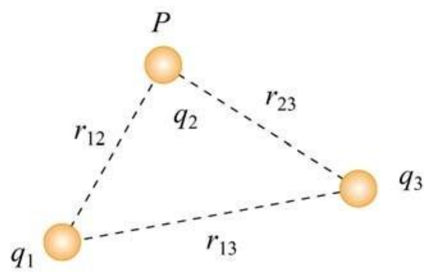


Figure 4.3 A system of three point charges

4.2 Electric Potential

Any charged particle located in a region of electrostatic field experiences a force. The force on the particle at any place is determined by the particle's charge and the value of the electric field: $F = qE$. If the particle moves from one place to another within that region, the electrostatic force does work on the particle and its potential energy changes.

The electric potential at any point in an electric field is equal to the work done against the electric field in moving a unit positive charge from outside the electric field to that point. It is a scalar quantity, as it is related to work done. The potential at a point is taken to be positive when work is done against the field by a positive charge. However, it is negative when work is done by the electric field in moving the unit positive charge from infinity to the point in the field. Therefore:

$$\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0} \tag{4.11}$$

It is assumed that $V(r=\infty)=0$. For a single point charge, the potential V is:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{4.12}$$

The S.I unit of electric potential is Volt (V) which is 1 Joule per Coulomb.

4.2.1 **Electric Potential due to a System of Charges**

For a total of N point charges, the potential V at any point P can be obtained from the principle of superposition. Recall that potential due to q_1 at point P is:

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} \qquad (4.13)$$

Therefore, total potential at point P due to N charges is:

$$V = V_1 + V_2 + \dots\dots + V_N \quad \text{(Principle of superposition)} \qquad (4.14)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \dots\dots\dots\dots + \frac{q_N}{r_N}\right] \qquad (4.15a)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i} \qquad (4.15b)$$

4.3 Equipotential Surfaces

For a given configuration of charges, a set of points where the electric potential $V(r)$ has a given value is called an equipotential surface. It is a surface on which the potential is constant. A charge can move freely on an equipotential surface without any work done, that is, $\Delta V = 0$ .This implies that no work can be done by the electric field to move an object along the surface, and thus $E.ds = 0$. Therefore, equipotential surfaces are always perpendicular to the direction of the electric field.

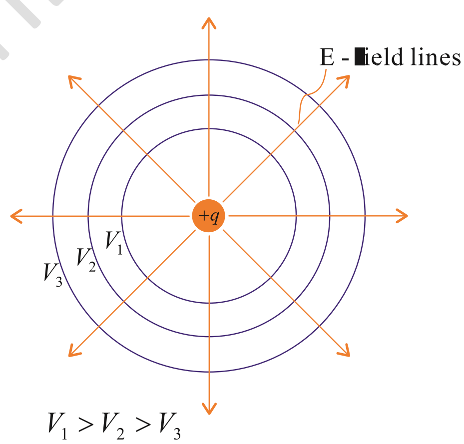


Figure 4.4. Equipotential surface (Equipotential surfaces are circles / spherical surfaces)

4.4 Relation Between Electric field and Electric Potential

(A) In order to derive the expression for electric potential V from electric field, we recall from the definition of change in electric potential that:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0} \tag{4.16}$$

Where ΔU is the change in electric potential energy; W_{12} is the work done in bringing charge q_0 from point 1 to point 2. Therefore,

$$\Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0} \tag{4.17}$$

But, from the definition of electric force: $\vec{F} = qE$

$$\therefore \Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s} \tag{4.18}$$

The integral on the right hand side of equation (4.18) can be calculated along any path from point 1 to point 2 (path- independent).

(B) To derive electric field **E** from electric potential V

Recall from the definition of V:

$$\Delta U = q_0 \Delta V = -W \tag{4.19}$$

But, $W = q_0 \vec{E}_s . \Delta s$ (Refer to Fig. 4.5) (4.20)

Where E_s is the electric field component along the path $\Delta \vec{s}$

$$\therefore q_0 \Delta V = -q_0 E_S \Delta s \tag{4.21}$$

$$\therefore E_s = -\frac{\Delta V}{\Delta s} \tag{4.22}$$

For infinitesimal Δs, gives:

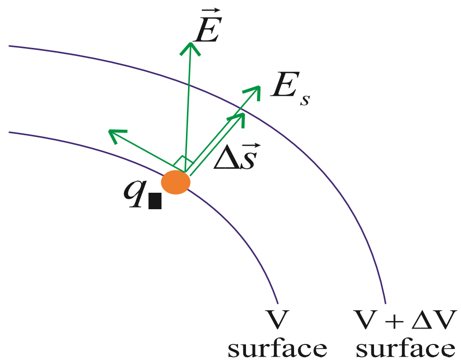
$$E_s = -\frac{dV}{dS} \tag{4.23}$$

Therefore, the electric field component along any direction is the negative derivative of the potential along the same direction.

Generally, for a potential $V(x, y, z)$, the relation between $E(x, y, z)$ and V is:

$$E_x = -\frac{\partial V}{\partial x} \quad , E_y = -\frac{\partial V}{\partial y} \quad , E_z = -\frac{\partial V}{\partial z} \tag{4.24}$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are partial derivatives.



(i.e. Potential = V on the surface)

Figure 4.5

Worked examples

4.1 The electric potential difference between the ground and a cloud in a particular thunderstorm is $1.2 \times 10^9\ V$. Determine the magnitude of the change in energy of an electron (in terms of electron-volt) that moves between the ground and the cloud.

Solution:

The magnitude of the change in potential as the electron moves between the ground and the cloud is:

$$|\Delta V| = 1.2 \times 10^9\ V$$

$$|U| = |q\,\Delta V| = e\left(1.2 \times 10^9\ V\right) = 1.2 \times 10^9\ V = 1.2\,G\,e\,V$$

4.2 Two large, parallel conducting plates are 12 cm apart and have charges of equal magnitude and opposite sign on their facing surfaces. An electrostatic force of 3.9 x 10^{-15} N acts on an electron placed anywhere between the two plates.

(a) Find the electric field at the position of the electron

(b) What is the potential difference between the plates?

Solution:

(a) The magnitude of the electric field is:

$$E = \frac{F}{|q|} = \frac{\left(3.9 \times 10^{-15}\ N\right)}{\left(1.60 \times 10^{-19}\ C\right)} = 2.4 \times 10^4\ N\,C^{-1} = 2.4 \times 10^4\ V\,m^{-1}$$

(b) The electric field in the region between two large oppositely- charged plates is uniform, therefore:

$$E_x = \frac{-\Delta V}{\Delta x}$$

The electric field points in the x-direction i.e perpendicular to the plates and the potential difference between the plates has magnitude:

$$|\Delta V| = |E_x \; \Delta x| = \left(2.4 \times 10^4 \; V\,m^{-1}\right)(0.12\; m) = 2.9 \times 10^3 \; V$$

4.3 A conducting sphere of radius 0.15 m has a potential difference of 200 V (with V = 0 at infinity). Calculate the: (a) charge (b) charge density on its surface.

Solution:

(a) Given that radius R of the conducting sphere = 0.15 m, to find the charge, we apply the Gauss's law. The electric field outside the sphere is the same as that of a point charge Q at the centre of the sphere. Therefore:

$$V = \frac{Q}{4\pi\,\varepsilon_0\,R} \Rightarrow Q = 4\pi\,\varepsilon_0\,V\,R = 4\pi\left(8.85 \times 10^{-12}\right)(200)(0.15)$$

$$Q = 3.3 \times 10^{-9}\; C$$

The charge on the sphere is $3.3 \times 10^{-9}\; C$

4.4 The electric potential at points in an x-y plane is given by:

$$V = \left(2.0\,V\,m^{-2}\right)x^2 - \left(3.0\,V\,m^{-2}\right)y^2 \quad .$$

Determine the magnitude and direction of the electric field at the point (3.0 m, 2.0 m).

Solution:

$$E_x = -\frac{\partial V}{\partial x} = -\left(4.0 V\,m^{-2}\right)x \quad \text{and} \quad E_y = -\frac{\partial V}{\partial y} = +\left(6.0\,V\,m^{-2}\right)y$$

By substituting the given values of $x = 3.0\; m$ and $y = 2.0\; m$, gives:

$$E_x = -12\,V\,m^{-1} \text{ and } E_y = -12\,V\,m^{-1}$$

The magnitude of the electric field is:

$$E = \sqrt{(12.0)^2 + (12.0)^2} = 17.0\,V\,m^{-1}$$

And its direction is given by:

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}(1.0)$$

$\theta = 135^0$ (θ lies in the second quadrant)

4.5 (a) Calculate the electric potential energy of two electrons that are separated by 2.00 nm.

(b) If this separation increases, does the potential energy increase or decrease?

Solution:

(a) Since the charge on an electron is negative, therefore:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{(-e)(-e)}{r} = \frac{1}{4\pi\left(8.85\times10^{-12}\right)}\frac{\left(1.60\times10^{-19}\right)^2}{\left(2.0\times10^{-9}\right)}$$

$$= 1.15\times10^{-19}\ J$$

(b) As the product of the charges is positive, the potential energy is a positive number and is inversely proportional to r. Therefore, potential energy decreases as r increases.

Self- Assessment Questions (SAQs)

1. An infinite non conducting sheet has a surface charge density $\sigma = 0.10\ \mu C m^{-2}$ on one side. How far apart are equipotential surfaces whose potential differs by 50 V ?
2. An empty hollow metal sphere has a potential of +400 V with respect to ground (defined to be at V = 0) and has a charge of $5.0\times10^{9}\ C$. Determine the electric potential at the centre of the sphere.
3. What is the excess charge on a conducting sphere of radius R = 0.15 m if the potential of the sphere is 1500 V and V= 0 at infinity?

SUMMARY

In this session, you have learnt that:

1. A force is said to be conservative if the work done on a particle by the force is independent of the path taken.

2. The change in potential energy associated with a conservative force F acting on an object as it moves from point 1 to 2 is:
$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}.d\vec{s} = -W$$
3. The electric potential at any point in an electric field is equal to the work done against the electric field in moving a unit positive charge from outside the electric field to that point. The change in electric potential ΔV is given by:
$$\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}$$
4. The electric potential due to a point charge q at a distance r away from the charge is:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

5. For a collection of charges, using the superposition principle, the electric potential is:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$$

6. From the electric potential V, the electric field may be obtained by taking the gradient of V:

$$E_x = -\frac{\partial V}{\partial x}\ ,\ E_y = -\frac{\partial V}{\partial y}\ ,\ E_z = -\frac{\partial V}{\partial z}$$

Study Session 5: Capacitance and Dielectrics

Expected Duration: 1 week or 2 contact hours

Introduction

A capacitor is a device that stores electric charge on their plates when connected to a voltage source. Capacitors vary in shape and size. It consists of two conductors carrying equal but opposite charges that are separated by a non-conducting material (an insulator). Capacitors have many important applications in electrons. Some examples include storing electric potential energy, delaying voltage changes when coupled with resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency- dependent and independent voltage dividers when combined with resistors. In this session, you will learn about the capacitance of a capacitor, different connections of capacitors, and the effects of dielectric materials on capacitance. Energy stored in a capacitor will also be discussed.

Learning Outcomes

When you have studied this session, you should be able to explain the:

5.1 Capacitance of a Capacitor

The simplest example of a capacitor consists of two conducting plates of area ***A***, which are parallel to each other, and separated by a distance *d*, as shown in Figure 5.1.

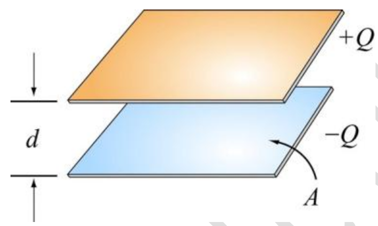


Figure 5.1: A parallel-plate capacitor

Experiments show that the amount of charge Q stored in a capacitor is linearly proportional to ΔV , the electric potential difference between the plates. Therefore,

$$Q = C\left|\Delta V\right| \tag{5.1}$$

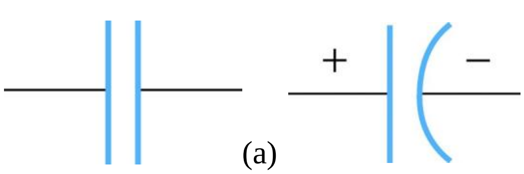
Where C is a positive proportionality constant called *capacitance.* Capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The ratio of the charge stored on the plates to the potential difference across the plates is called the capacitance C of the capacitor. The dimensions of capacitance are $M^{-1}L^{-2}T^{2}Q^{2}$. The S.I unit of capacitance is the *farad* (F).

$$1\text{ F} = 1\text{ farad} = 1\text{ coulomb/volt} = 1\text{ C/V}$$

A typical capacitance is in the picofarad (1 pF $=10^{-12}$ F) to millifarad range,

(1 mF $=10^{-3}$ F=1000μF; 1μF $=10^{-6}$ F).

Figure 5.2(a) shows the symbol which is used to represent capacitors in circuits. For a polarized fixed capacitor which has a definite polarity, Figure 5.2(b) is sometimes used.



(a) (b)

Figure 5.2: Capacitor symbols.

For a parallel-plate capacitor with plates of area *A* separated by distance *d*, the capacitance is given by:

$$C = \frac{\varepsilon_0 A}{d} \tag{5.2}$$

Equation (5.2) shows that C depends only on the geometric factors *A* and *d.* The capacitance C increases linearly with the area *A* since for a given potential difference ΔV, a bigger plate can hold more charge. On the other hand, C is inversely proportional to *d*, the distance of separation because the smaller the value of *d*, the smaller the potential difference $|\Delta V|$ for a fixed *Q*.

- **Cylindrical Capacitor**

In this geometry, there are two coaxial cylinders where the radius of the inner conductor is *a* and the inner radius of the outer conductor is *b*. The length of the cylinders is *L*; (*L* is large compared to *b*). For this geometry, the capacitance is given by:

$$C = 2\pi\varepsilon_0 \frac{L}{In\left(\frac{b}{a}\right)} \tag{5.3}$$

- **Spherical Capacitor**

In this geometry, there are two concentric spheres where the radius of the inner sphere is *a* and the inner radius of the outer sphere is *b*. For this geometry, the capacitance is given by:

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \tag{5.4}$$

Again, the capacitance C depends only on the physical dimensions, *a* and *b*.
An "isolated" conductor (with the second conductor placed at infinity) also has a capacitance. In the limit where $b \rightarrow \infty$, the above equation becomes:

$$\lim_{b\rightarrow\infty} C = 4\pi\varepsilon_0 a \tag{5.5}$$

Thus, for a single isolated spherical conductor of radius R, the capacitance is:

$$C = 4\pi\varepsilon_0 R \tag{5.6}$$

5.2 Combination of Capacitors

5.2.1 Capacitors in Series

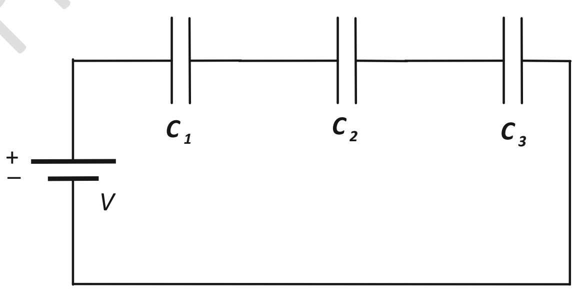


Figure 5.3: Three capacitors are combined in series across a potential difference V

Figure 5.3 shows a configuration where three capacitors are combined in series across the terminals of a battery. The charge is the same on each capacitor and the potential difference across the system is the sum of the potential difference across the individual capacitance. Therefore:

$$V = V_1 + V_2 + V_3 \tag{5.7}$$

But $V_1 = \dfrac{q}{C_1} \qquad V_2 = \dfrac{q}{C_2} \qquad V_3 = \dfrac{q}{C_3}$ (5.8)

$$\therefore \; \frac{q}{C} = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} \tag{5.9}$$

This gives:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{5.10}$$

The generalization to any number of capacitors connected in series is:

$$\frac{1}{C_{eq.}} = \frac{1}{C_1} + \frac{1}{C_2} + \Lambda + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i} \tag{5.11}$$

5.2.2 Capacitors in Parallel

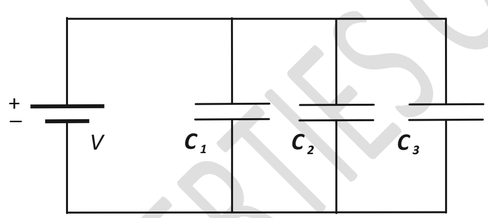


Figure 5.4: Three capacitors are combined in parallel across a potential difference difference V

Figure 5.4 shows a configuration where three capacitors are combined in parallel across the terminals of a battery. The battery gives a constant potential difference V across the plates of each of the capacitors. The total charge is the sum of the charges on the individual capacitor. Therefore:

$$q = q_1 + q_2 + q_3 \tag{5.12}$$

But $q_1 = C_1 V \quad q_2 = C_2 V \quad q_3 = C_3 V$ (5.13)

$$\therefore \; CV = C_1 V + C_2 V + C_3 V \tag{5.14}$$

$$C = C_1 + C_2 + C_3 \tag{5.15}$$

In general, the equivalent capacitance for a set of capacitors which are in connect in parallel is given by:

$$C_{eq} = C_1 + C_2 + C_3 + \Lambda + C_N = \sum_{i=1}^{N} C_i \qquad (5.16)$$

5.3 Dielectrics

A dielectric is an insulator or non-conductor of electricity. In many capacitors there is an insulating material such as paper or plastic between the plates. Such material, called a dielectric, can be used to maintain a physical separation of the plates. Its effects are defined in terms of the ratio of the capacitance of a capacitor with a dielectric between the plates to the capacitance of a capacitor without a dielectric between the plates. Experimentally, it was found that capacitance C increases when the space between the conductors is filled with dielectrics. This can be explained by considering the fact that when there is no material between the plates, a capacitor is assumed to have a capacitance C_0. When a dielectric is inserted to completely fill the space between the plates, the capacitance increases to

$$C = K_e C_0 \qquad (5.17)$$

Where K_e is called the dielectric constant.

5.4 Energy Stored in a Capacitor

The work required to charge up a capacitor by moving a charge $-q$ from one plate to another is the potential energy U of the charges. This potential energy is the energy stored in the electric field between the plates of the capacitor. This energy is given by:

$$U = \frac{q^2}{2C} = \frac{1}{2} C V^2 \qquad (5.18)$$

5.4.1 Energy Density of the Electric Field

The energy stored in the capacitor can be considered as being stored in the electric field itself. In the case of a parallel-plate capacitor with $C = \varepsilon_0 \frac{A}{d}$ and $|\Delta V| = E\,d$, then the energy is:

$$U_E = \frac{1}{2} C |\Delta V|^2 = \frac{1}{2} \varepsilon_0 \frac{A}{d} (E\,d)^2 = \frac{1}{2} \varepsilon_0 E^2 (Ad) \qquad (5.19)$$

The quantity Ad represents the volume between the plates, we can define the electric energy density as:

$$u_E = \frac{U_E}{Volume} = \frac{1}{2} \varepsilon_0 E^2 \qquad (5.20)$$

In equation (5.4.3), u_E is proportional to the square of the electric field.

Worked Examples

5.1 A parallel-plate capacitor has square plates 7.5cm on a side, separated by 0.29mm. The capacitor is charged to 12V, then disconnected from the charging power supply.

(a) Calculate the capacitance of this capacitor.

(b) What is the total charge on each plate? What is the charge density on the plates?

(c) What is the electric field between the plates?

Solution:

(a) The capacitance for a parallel-plate capacitor is given by:

$$C = \varepsilon_0 \frac{A}{d} = \varepsilon_0 \frac{l^2}{d}$$

$$C = \left(8.85 \times 10^{-12}\ F\,m^{-1}\right)\frac{(0.075\,m)^2}{\left(2.9 \times 10^{-4}\ m\right)^2} = 1.7 \times 10^{-10}\ F$$

(b) 1. The total charge on each plate is:

$$Q = CV = \frac{\varepsilon_0 l^2 V}{d}$$

$$= \frac{\left(8.85 \times 10^{-12}\ F\,m^{-1}\right)(0.075\,m)^2 (12V)}{\left(2.9 \times 10^{-4}\ m\right)} = 2.1 \times 10^{-9}\ C$$

(b)2. The charge density is the charge on the plate divided by the plate's area:

$$\sigma = \frac{Q}{A} = \frac{CV}{l^2} = \frac{\varepsilon_0 V}{d}$$

$$= \frac{\left(8.85 \times 10^{-12}\ F\,m^{-1}\right)(12V)}{2.9 \times 10^{-4}\ m} = 3.7 \times 10^{-7}\ C\,m^{-2}$$

(c) The electric field between two oppositely charged plates is:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{V}{d} = \frac{(12\,V)}{\left(2.9 \times 10^{-4}\ m\right)} = 4.1 \times 10^{4}\ V\,m^{-1}$$

5.2 Consider a parallel-plate capacitor in which the separation between its plates can be varied. The maximum capacitance it can withstand is 120 pF.The capacitor is charged to a potential difference of 50 mV at maximum capacitance and then isolated. With the capacitor still isolated, what is the distance between its plates so that it now has a potential difference of 30 V? The area of the plate is 3.1 cm^2.

Solution:

Since the capacitor is kept in isolation, the charge on the plates is the same after the plate separation is changed. Therefore:

$$Q_i = Q_f \Rightarrow C_i V_i = C_f V_f$$

$$C_f = \frac{C_i V_i}{V_f}$$

Using the equation for the capacitance of a parallel-plate capacitor, the plate separation d_f is:

$$d_f = \frac{\varepsilon_0 A}{C_f} = \frac{\varepsilon_0 A V_f}{C_i V_i} = \frac{\left(8.85\times10^{-12}\ F\,m^{-1}\right)\left(3.1\times10^{-4}\ m^2\right)\left(30V\right)}{\left(120\times10^{-12}\ F\right)\left(50\times10^{-3}\ V\right)}$$

$$= 1.4\ cm$$

5.3 Calculate the energy stored in one cubic metre of air due to the "fair weather" electric field of magnitude 150 V/m.

Solution:

Energy density of an electric field, $u_E = \frac{1}{2}\varepsilon_0 E^2$

$$= \frac{1}{2}\left(8.85\times10^{-12}\ F\,m^{-1}\right)\left(150\,V\,m^{-1}\right)^2 = 9.96\times10^{-8}\ J\,m^{-3}$$

Therefore, in one cubic metre, $9.96\times10^{-8}\ J$ of energy is stored.

5.4 A $2.0-\mu F$ spherical capacitor is composed of two metal spheres, one having a radius twice as large as the other. If the region between the spheres is a vacuum, determine the volume of this region.

Solution:

The capacitance of a spherical capacitor is:

$$C = 4\pi\varepsilon_0 \frac{ab}{(b-a)}$$

Where a and b are the radii of the concentric spherical plates. Given that: b = 2a

$$C = 4\pi\varepsilon_0 \frac{2a^2}{a} = 8\pi\varepsilon_0 a$$

$$\therefore\ a = \frac{C}{8\pi\varepsilon_0} = \frac{\left(2.0\times10^{-6}\ F\right)}{8\pi\left(8.85\times10^{-12}\ F\,m^{-1}\right)} = 9.0\times10^3\ m$$

So that $b = 2a = 1.8\times10^4\ m$

Therefore, the volume of the enclosed region between the two plates is:

$$V_{enc.} = \frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3 = \frac{4}{3}\pi\left((2a)^3 - (a)^3\right) = \frac{4}{3}\pi\left(7a^3\right)$$

$$\frac{4}{3}\pi\,(7)\left(9.0\times10^3\ m\right)^3 = 2.1\times10^{13}\ m^3$$

The volume of the enclosed region is $2.1\times10^{13}\ m^3$

5.5 A parallel-plate air filled capacitor has a capacitance of $50\ pF$.

(a) If each of its plates has an area of $0.35m^2$, what is the separation?

(b) If the region between the plates is now filled with a material having $k = 5.6$, what is the capacitance?

Solution:

(a) Given that: $C = 50\ pF$, $A = 0.35m^2$ $k = 5.6$

$$C = \frac{\varepsilon_0\ A}{d} \Rightarrow d = \frac{\varepsilon_0\ A}{C} = \frac{(8.85\times10^{-12}\ F\,m^{-1})(0.35\,m)^2}{(50\times10^{-12}\ F)}$$

$$d = 6.2\times10^{-3}\ m$$

(b) With a dielectric between the plates, the new capacitance is given by:

$$C_{new} = \frac{(k\,\varepsilon_0)A}{d} = k\left(\frac{\varepsilon_0\ A}{d}\right) = k\,(C_{old})$$

$$C_{new} = (5.6)(50\ pF) = 280\ pF$$

Self –Assessment Questions (SAQs)

1. Two capacitors, of $2.0\ \mu F$ and $4.0\ \mu F$ capacitance, are connected in parallel across a $300\,V$ potential difference. Calculate the total energy stored in the capacitors.
2. A parallel- plate capacitor has a circular plates of $8.20\ cm$ radius and $1.30\ mm$ separation.
   (a) Calculate the capacitance
   (b) Find the charge for a potential difference of $120\,V$
3. A $100\ pF$ capacitor is charged to a potential difference of $50\ V$, and the charging battery is disconnected. The capacitor is then connected in parallel with a second (initially charged). If the potential difference across the first capacitor drops to $35\ V$, what is the capacitance of this second capacitor?
4. The space between two concentric conducting spherical shells of radii $a = 1.70\ cm$ and $b = 1.20\ cm$ is filled with a substance of dielectric constant $k = 23.5$. A potential difference of 73.0 volts is applied across the inner and outer shells. Determine the:
   (a) Capacitance of the capacitor
   (b) Free charge on the inner shell
   (c) The charge ‘q” induced along the surface of the inner shell

5. Two parallel plates of area $100\ cm^2$ are given charges of equal magnitude $8.9\times10^{-7}\ C$ but of opposite sign. The electric field within the dielectric material filling the space between the plates is $1.4\times10^{6}\ V\,m^{-1}$.
   (a) Calculate the dielectric constant of the material
   (b) Determine the magnitude of the charge induced on each dielectric surface.

SUMMARY

In this session, you have learnt that:

1. Capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The ratio of the charge stored on the plates to the potential difference across the plates is called the capacitance C of the capacitor
2. Capacitance of a capacitor depends on its shape, size and nature of medium.
3. The capacitance of a dielectric filled parallel-plate capacitor is K times the capacitance with vacuum or air as dielectric.
4. The potential energy stored in the electric field between the plates of a capacitor is given by:

$$U=\frac{q^2}{2C}=\frac{1}{2}CV^2$$

5. In series combination of capacitors, the charge is the same on each capacitor and the potential difference across the system is the sum of the potential difference across the individual capacitance. The generalization to any number of capacitors connected in series is:

$$\frac{1}{C_{eq.}}=\frac{1}{C_1}+\frac{1}{C_2}+\Lambda+\frac{1}{C_N}=\sum_{i=1}^{N}\frac{1}{C_i}$$

6. In parallel combination of capacitors, the total charge is the sum of the charges on the individual capacitor. The same potential difference V is maintained across the plates of

each of the capacitors. The generalization to any number of capacitors connected in parallel is:

$$C_{eq} = C_1 + C_2 + C_3 + \Lambda + C_N = \sum_{i=1}^{N} C_i$$

Study Session 6: Current Electricity

Expected Duration: 1 week or 2 contact hours

Introduction

Electric current is the flow of charged particles. In this session, you will learn about the properties of electric currents, ohm's law, drift velocity and temperature dependence on resistance.

Learning Outcomes

When you have studied this session, you should be able to explain the:

6.1 Electric current and Current Density

6.2 Resistance, Resistivity and Conductivity

6.3 Electromotive Force and Internal Resistance

6.1.1 Electric current

The flow of electric charges constitute an electric current. The moving electric charges may either be electrons or ions. Therefore, the current in metal wire is due to the flow of electrons. Whenever there is a net flow of charge through some region, an electric current is said to exist. If a net amount of charge ΔQ flows perpendicularly through a particular surface of area A within a time interval Δt, the electric current I is:

$$I = \frac{\Delta Q}{\Delta t} \tag{6.1}$$

The unit of electric current is Ampere (A) or Coulombs per second (C/s). Therefore, 1 A of current is equivalent to 1C of charge passing through the surface in 1 s. Electric current has an associated direction. By convention, the direction of current flow is in the direction of positive charge movement as shown in Figure 6.1.The electrons move in the opposite direction.

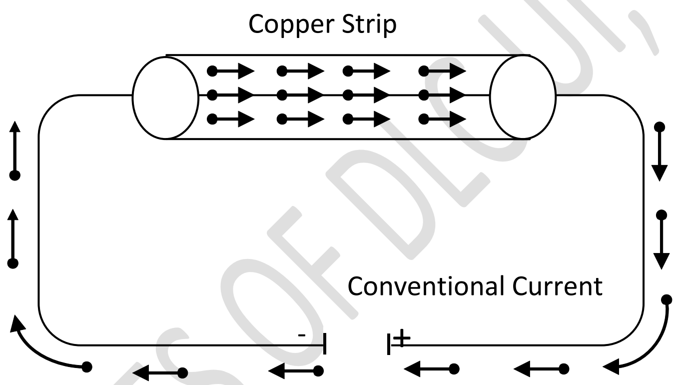


Fig. 6.1 Flow of Electric Current

Therefore, the direction of the current is opposite to the direction of the flow of electrons. A moving charge, positive or negative, is usually referred to as a mobile charge carrier.

6.1.2 Electric Current Density

The current across an area can be expressed in terms of the motion of the charge carrier. To achieve this, let us consider a portion of a conductor of cross-sectional area A as shown in figure 6.2. The volume element for a length Δx is $A\Delta x$. If n is the number of electrons per unit volume, the number of electrons in this volume element will be $nA\Delta x$. The total charge in this volume element is $\Delta q = (nA\Delta x)e$, where e is charge on the electron. If electron drift with a speed V_d due to thermal energy, the distance travelled in time Δt is $\Delta x = V_d \, \Delta t$. On substituting this value of Δx in the expression for Δq, we find that total charge in the volume element under consideration is given by:

$$\Delta q = nAeV_d \, \Delta t \tag{6.2}$$

Therefore, $$I = \frac{\Delta q}{\Delta t} = nAeV_d \tag{6.3}$$

The electric current density $\vec{J}$ is a vector quantity whose magnitude is the ratio of the magnitude of electric current flowing in a conductor to the cross-sectional area perpendicular to the current flow and whose direction points in the direction of the current.

$$\vec{J} = \frac{I}{A} \tag{6.4}$$

Using the relation $I = nAeV_d$, gives:

$$\vec{j} = ne\vec{V}_d \tag{6.5}$$

The S.I unit of current density is Am^{-2}. Equation (6.5) is valid only if J is uniform and the direction of *I* is perpendicular to the cross-sectional area A. The amount of current that passes through an element of area dA, can be written as $\vec{J} \cdot d\vec{A}$, where $d\vec{A}$ is the vector area of the element. The current that passes throughout the entire area A is then:

$$I = \int \vec{J} \cdot d\vec{A} \tag{6.6}$$

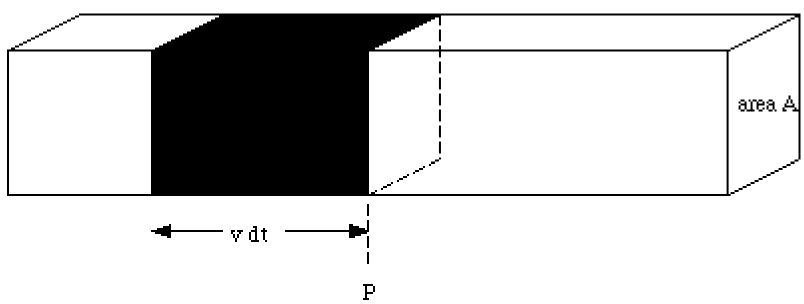


Figure 6.2 Motion of electron in conductor

6.2 Resistance, Resistivity and Conductivity

When a potential difference ΔV is maintained across a conductor, an electric field $\vec{E}$ and a current density $\vec{J}$ are established in the conductor. The current density $\vec{J}$ is directly proportional to the electric field, $\vec{E}$ at a given temperature for some materials with electrical properties that are the same in all directions. That is:

$$\vec{E} = \rho \vec{J} \tag{6.7}$$

Where the constant ρ is called the resistivity of the conductor. Materials that obey this relation are said to obey ohm's law and are called ohmic materials. For ohmic materials, the resistivity at a given temperature is nearly constant. If a material does not obey ohm's law, the material is called non-ohmic material for example, semiconductor. An equivalent form of ohm's law given by equation (6.7) is derived as follows:

Recall that for uniform electric field, we have:

$$\Delta V = E\,L \quad \Rightarrow \frac{\Delta V}{L} = E = \rho\, J \tag{6.8}$$

But $J = \dfrac{i}{A} \Rightarrow \Delta V = \rho\left(\dfrac{i}{A}\right)L = \left(\dfrac{\rho L}{A}\right)i$ (6.9)

The quantity in bracket is called the electrical resistance of the conductor.

$$R = \frac{\rho L}{A} \tag{6.10}$$

$\Delta V = i\,R$ (Equivalent form of Ohm's law) (6.11)

Therefore, the resistance of a conductor is defined as the ratio of potential difference across the conductor to the current flowing through it. The unit of resistance is ohm (Ω). The reciprocal of resistance is conductance and its unit is mho (Ω^{-1}). The resistance, R is directly proportional to its length and is inversely proportional to its area of cross-section. The electrical resistivity of a material is defined as the resistance offered to current flow by a conductor of unit length having unit area of cross-section. The unit of electrical resistivity, ρ is ohm-metre (Ωm). It is constant for a particular material. The reciprocal of electrical resistivity is called conductivity. Ohm's law states that, at a constant temperature, the steady current flowing through a conductor is directly proportional to the potential difference between the two ends of the conductor.

6.2.1 Variation of Resistance with Temperature

The resistivity of substances varies with temperature. For conductors, the resistance increases with increase in temperature. If R_0 is the resistance of a conductor at a reference temperature and R_θ is the resistance of some conductors at $\theta^0 C$. Thus:

$$R_\theta = R_0\left[1 + \alpha\left(\theta - \theta_0\right)\right] \tag{6.12}$$

Since, $R \propto \rho \quad \Rightarrow \rho_\theta = \rho_0\left[1 + \alpha\left(\theta - \theta_0\right)\right]$ (6.13)

Where ρ_θ = resistivity at temperature θ

$\rho_0 =$ Resistivity at a reference temperature

$\alpha =$ Temperature coefficient of resistivity

The temperature coefficient of resistivity is defined as the ratio of increase in resistance per degree rise in temperature to its resistance at a reference temperature. Its unit is per ^0C . Metals have positive temperature coefficient because their resistance increase with increase in temperature. Semiconductors and insulators have negative temperature coefficient since their resistance decreases with increase in temperature.

6.3 Electromotive Force and Internal Resistance

To maintain a steady current in an external circuit, a battery (cells) is needed to supply electrical energy. The electric current in an external circuit flows from the positive terminals to the negative terminal of the cell through different circuit elements. Figure 6.3 shows the two terminals of a battery. Because of the positive and negative charges on the battery terminals, an electric potential difference exits between them. In moving from point 1 to 2, electric potential energy increases by

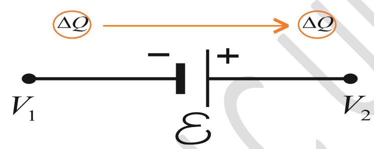


Figure 6.3: Battery terminals

$$\Delta U = \Delta Q(V_2 - V_1) = E \tag{6.14}$$

$$E = \frac{work\,done}{charge} \tag{6.15}$$

The maximum potential difference is called the electromotive force, E (or emf), of the battery. The emf of a source is defined as the amount of work done or energy required to move a unit positive charge from the low-potential terminal to the high–potential terminal. The S.I unit of emf is Volt (V). The different sources of emf are: batteries, electric generators, fuel cells and solar cells. The electromotive force of a battery or other electric power source. A source with a time-independent emf is represented by the symbol shown in Figure 6.4. The long line in Figure 6.4 represents the positive terminal of the source, while the short line represents the negative terminal.

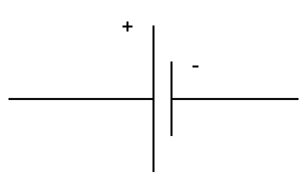


Figure 6.4 Schematic Symbol of Source of emf

The potential difference between two electrodes of a cell when it is in closed circuit, that is, when current is drawn from the cell is called terminal potential difference. It is denoted by V_T. The potential difference between the positive and negative electrodes in an open circuit, that is, when no current is drawn from the cell is called the emf of the cell. The main current that flows in a simple circuit shown in Figure 6.5 can be determined by using the Ohm's law.

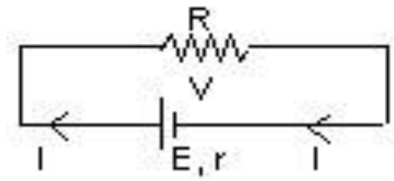


Figure 6.5. A simple circuit diagram with external resistance, emf source and internal resistance r.

Consider an external resistance 'R' connected across the cell. Let I, be the current that flows in the circuit. 'E' is the emf of the cell, r is the internal resistance of the cell. 'V' is the potential difference across R. When the resistance, R, is infinite, then there is no current in the circuit (open circuit). The potential difference across is

$V = E = V_+ + V_-$ (terminal potential difference)

If 'R' is finite, the current is not equal to zero. The potential difference between ends of the cell is

$$V = V_+ + V_- - I\,r$$

$$V = E - I\,r \qquad (6.16)$$

r is the internal resistance of the cell. It is the finite resistance offered by the electrolyte for the flow of current through it. The negative sign in the expression $(I\,r)$ indicates that the direction of current is in opposite direction in the electrolyte.

$$I\,R + I\,r = E \;\Rightarrow\; I = \frac{E}{R + r} \qquad (6.17)$$

The maximum current that can be drawn from a cell is $I_{\max} = \dfrac{E}{r}$ $(R = 0)$.

The internal resistance of a cell depends on:

(1)The nature of electrolyte (2) nature of electrodes (3) temperature
(4) concentration of electrodes (5) distance between the electrodes

Worked Examples

6.1 The number density of free electrons in a copper conductor is $8.5 \times 10^{28}\ m^{-3}$. How long does an electron take to drift from one end of a wire $3.0\ m$ to its other end? The area of cross-section of the wire is $2.0 \times 10^{-6}\ m^2$ and it is carrying a current of $3.0\ A$.

Solution:

Number density of free electrons in a copper conductor, $n = 8.5 \times 10^{28}\ m^{-3}$.

Length of the copper wire $l = 3.0\ m$

Area of cross- section of the wire, $A = 2.0 \times 10^{-6}\ m^2$

Current carried by the wire $I = 3.0\ A$

Using $I = nAeV_d$

$$V_d = drift\ velocity\ = \frac{length\ of\ the\ wire\,(l)}{Time\ taken\ to\ cover\ l\ (t)}$$

$$I = nAe\left(\frac{l}{t}\right) \Rightarrow t = \frac{nAel}{I} = \frac{\left(8.5 \times 10^{28}\ m^{-3}\right)\left(2.0 \times 10^{-6}\ m^2\right)\left(1.6 \times 10^{-19}\ C\right)(3.0\ m)}{(3.0\ A)}$$

$$= 2.7 \times 10^4\ s$$

Therefore, the time taken by an electron to drift from one end of the wire to the other end is $2.7 \times 10^4\ s$.

6.2 A silver wire has a resistance of $2.1\,\Omega$ at $27.5^0 C$, and a resistance of $2.7\,\Omega$ at $100^0 C$. Determine the temperature coefficient of resistivity of silver.

Solution:

Temperature, $T_1 = 27.5^0 C$, Resistance of the silver wire at T_1, $R_1 = 2.1\,\Omega$

Temperature $T_2 = 100^0 C$, Resistance of the silver wire at T_2 , $R_2 = 2.7\,\Omega$

Temperature coefficient of silver, $\alpha = \dfrac{R_2 - R_1}{R_1(T_2 - T_1)} = \dfrac{2.7 - 2.1}{2.1(100 - 27.5)} = 0.0039^0 C^{-1}$

Therefore, the temperature coefficient of silver is $0.0039^0 C^{-1}$.

6.3 Two wires of equal length, one is aluminium and the other is copper have the same resistance. Which of the two wires is lighter? Hence explain why aluminium wire is preferred for overhead power cables.

$\left(\rho_{Al} = 2.63 \times 10^{-8}\ \Omega m,\ \rho_{cu} = 1.72 \times 10^{-8}\ \Omega m,\ Relative\ density\ of\ Al = 2.7\ \right)$

$Relative\ density\ of\ Cu = 8.9$

Solution:

Resistivity of aluminium, $\rho_{Al} = 2.63 \times 10^{-8}\ \Omega m$

Resistivity density of aluminium, $d_1 = 2.7$

Let l_1 be the length of aluminium wire and m_1 be its mass

Resistance of the aluminium wire $= R_1$

Area of cross-section of the aluminium wire $= A_1$

Resistivity of copper, $\rho_{Cu} = 1.72 \times 10^{-8}\ \Omega m$

Relative density of copper, $d_2 = 8.9$

Let l_2, be the length of copper wire and m_2 be its mass

Resistance of the copper wire $= R_2$

Area of cross-section of the copper wire $= A_2$

Using $R_1 = \dfrac{\rho_1 l_1}{A_1}$ (1)

$$R_2 = \frac{\rho_2 l_2}{A_2} \quad (2)$$

Given that $R_1 = R_2 \Rightarrow \dfrac{\rho_1 l_1}{A_1} = \dfrac{\rho_2 l_2}{A_2}$

Since, $l_1 = l_2 \Rightarrow \dfrac{A_1}{A_2} = \dfrac{\rho_1}{\rho_2} = \dfrac{2.63 \times 10^{-8}\ \Omega m}{1.72 \times 10^{-8}\ \Omega m} = \dfrac{2.63}{1.72}$

Mass of the aluminium wire, $m_1 = Volume \times Density = A_1 l_1 d_1$ (3)

Mass of the copper wire, $m_2 = Volume \times Density = A_2 l_2 d_2$ (4)

Dividing equation (3) by equation (4) gives:

$$\frac{m_1}{m_2} = \frac{A_1 l_1 d_1}{A_2 l_2 d_2}$$

Since $l_1 = l_2$ $\dfrac{m_1}{m_2} = \dfrac{A_1 d_1}{A_2 d_2} = \dfrac{2.63}{1.72} \times \dfrac{2.7}{8.9} = 0.46$

It can be inferred from this ratio that m_1 is less than m_2. Hence, aluminium is lighter than copper.

6.4 A heating element using nichrome connected to a 230 V supply draws an initial current of 3.2 A which settles after a few seconds to a steady value of 2.8 A. What is the steady temperature of the heating element if the room temperature is $27^0 C$? Temperature co-efficient of resistance of nichrome averaged over the temperature ranged involved is $1.70 \times 10^{-4}\ {}^0C^{-1}$.

Solution:

Supply voltage, $V = 230\ volts$

Initial current drawn, $I_1 = 3.2\ A$

Initial resistance, $R_1 = \dfrac{V_1}{I_1} = \dfrac{230}{3.2} = 71.87\ \Omega$

Steady state value of the current, $I_2 = 2.8\ A$

Resistance at the steady state R_2 is given by:

$$R_2 = \frac{V_1}{I_2} = \frac{230}{2.8} = 82.14\ \Omega$$

Temperature co-efficient of nichrome, $\alpha = 1.70 \times 10^{-4}\ {}^0C^{-1}$

Initial temperature of nichrome, $T_1 = 27.0\ {}^0C$

Steady state temperature reached by nichrome $= T_2$

Using $\alpha = \dfrac{R_2 - R_1}{R_1(T_2 - T_1)} \Rightarrow T_2 - T_1 = \dfrac{R_2 - R_1}{\alpha R_1} = \dfrac{82.14 - 71.87}{(1.70 \times 10^{-4})(71.87)} = 840.5$

$T_2 = 867.5^0 C$

6.5 A battery of emf of 10V having an internal resistance of 3Ω is connected to a resistor.

(a) If the current in the circuit is 0.5 A, determine the resistance of the resistor.

(b) Calculate the terminal voltage of the battery when the circuit is closed.

Solution:

(a) Emf of the battery, $E = 10\,V$ Internal resistance of the battery, $r = 3\Omega$

Current in the circuit, $I = 0.5\,A$, Resistance of the resistor $= R$

The relation for current using ohm's law is:

$$I = \frac{E}{R+r} \Rightarrow R + r = \frac{E}{I} = \frac{10}{0.5} = 20\,\Omega$$

$$\therefore R = (20-3)\Omega = 17\,\Omega$$

(b) Terminal voltage of the resistor $= V$

According to ohm's law, $V = I\,R = 0.5 \times 17 = 8.5\,V$

Therefore, the resistance of the resistor is $17\,\Omega$ and the terminal voltage is $8.5\,V$

6.6 A beam of electrons moving at a speed of $10^6\ ms^{-1}$ along a line produces a current of $1.6 \times 10^{-6}\,A$. Calculate the number of electrons in the 1 metre of the beam.

Solution:

Using $i = \dfrac{q}{t} = \dfrac{q}{\left(\dfrac{x}{v}\right)} = \dfrac{qv}{x} = \dfrac{nev}{x} \Rightarrow n = \dfrac{ix}{ev} = \dfrac{(1.6 \times 10^{-6})(1)}{(1.6 \times 10^{-19})(10^6)} = 10^7$ electrons

Self- Assessment Questions (SAQs)

1. The storage battery of a car has an emf of 12V. If the internal resistance of the battery is $0.4\,\Omega$, what is the maximum current that can be drawn from the battery?
2. A coil of wire has a resistance of $40\,\Omega$ at 25^0C. Determine its resistance at 55^0C ,if the temperature coefficient of the material is 0.0043^0C^{-1} at 0^0C.
3. A copper wire of length 1 m and radius 1 mm is joined in series with an iron wire of length 2m and radius 3 mm and a current is passed through the wire. Calculate the ratio of current densities in the copper and iron wire.
4. A conducting wire of cross-sectional area $1\,cm^2$ has $3 \times 10^{23}\ m^{-3}$ charge carrier. If the wire carries a current of $24\ mA$, determine the drift speed of the carrier.
5. The resistance of a wire at 20^0C is 20Ω and at 500^0C is 60Ω. At what temperature will the resistance be 25Ω?
6. Masses of three wires are in the ratio 1:3:5 and their lengths are in the ratio 5:3:1. Determine the ratio of their electrical resistance.

7. A rod of certain metal is $1\,m$ long and $0.6\,cm$ in diameter. Its resistance is $3\times10^{-3}\ \Omega$. A disc of the same metal is $1\ mm$ thick and $2\ cm$ in diameter. Calculate the resistance between its circular faces.

SUMMARY

In this session, you have learnt that:

1. The flow of electric charges constitute an electric current. The unit of electric current is Ampere (A) or Coulombs per second (C/s).
2. The electric current density $\vec{J}$ is a vector quantity whose magnitude is the ratio of the magnitude of electric current flowing in a conductor to the cross-sectional area perpendicular to the current flow and whose direction points in the direction of the current. The S.I unit of current density is Am^{-2}.
3. The resistance of a conductor is defined as the ratio of potential difference across the conductor to the current flowing through it. The unit of resistance is ohm. The electrical resistivity of a material is defined as the resistance offered to current flow by a conductor of unit length having unit area of cross-section. The unit of electrical resistivity, ρ is ohm-metre (Ωm).
4. Ohm's law states that, at a constant temperature, the steady current flowing through a conductor is directly proportional to the potential difference between the two ends of the conductor.
5. The electromotive force (emf) of a source is defined as the amount of work done or energy required to move a unit positive charge from the low-potential terminal to the high–potential terminal. The S.I unit of emf is Volt (V)
6. The internal resistance of a cell is the infinite resistance offered by the electrolyte for the flow of current through it.

Study Session 7: Analysis of Direct – Current Circuits Containing Resistors and Kirchhoff's Laws

Expected Duration: 1 week or 2 contact hours

Introduction

The motion of electric charges creates electric current. In session 6, you learnt about the properties of electric current and ohm's law. In this session, you will learn about the analysis of simple electric circuit that contains batteries, capacitors and resistors in various combinations. The analysis of more complicated circuits is simplified using Kirchhoff's which results from the laws of conservation of energy and conservation of electric charges for isolated systems. Electrical energy and power will also be discussed.

Learning Outcomes

When you have studied this session, you should be able to explain the:

7.1 Effective Resistance for a Series and Parallel Combination of Resistors

7.2 Kirchhoff's Rules to Closed Electrical Circuits

7.3 Electrical Energy and Power

7.1 Effective Resistance for a Series and Parallel Combination of Resistors

7.1.1 Series Combination

Two or more resistors are said to be connected in series when they are connected in such a way that the same current flows through all these resistors. Let's consider three resistors R_1 , R_2 and R_3 connected in series and a potential difference of "V" volts is applied across it as shown in figure 7.1. In this circuit, the current through each resistor will be the same but the potential difference across each is different. Using Ohm's law:

Potential difference across R_1 , $V_1 = I R_1$

Potential difference across R_2 , $V_2 = I R_2$

Potential difference across R_3 , $V_3 = I R_3$

If 'V' is the effective potential drop and R_T is the effective resistance, then the effective potential difference across the combined resistors is:

$$V = V_1 + V_2 + V_3 \tag{7.1}$$

$$I R_T = I R_1 + I R_2 + I R_3 \tag{7.2}$$

$$R_T = R_1 + R_2 + R_3 \tag{7.3}$$

Therefore, the effective resistance of the series combination of a number of resistors is equal to the sum of the resistances of each resistor.

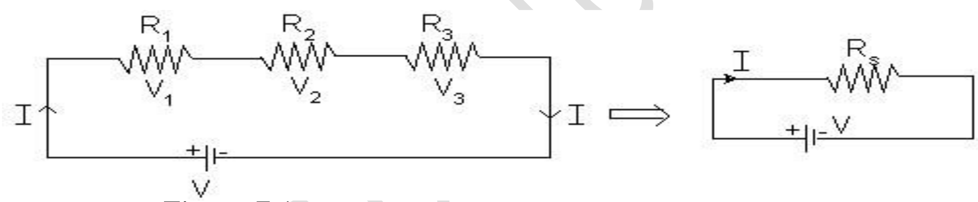


Figure 7.1

7.1.2 Parallel Combination

A number of resistors are said to be connected in parallel when they are connected in such a way that the same potential difference is maintained across each of the resistors. Consider three resistors R_1, R_2 and R_3 are connected in parallel across a potential difference of V as shown in Figure 7.2. In Figure 7.2, all the resistors are connected across the same terminal, the potential difference across all the resistors are equal, the current in each resistor is given by:

$$I_1 = \frac{V}{R_1} \ , \ I_2 = \frac{V}{R_2} \ , \ I_3 = \frac{V}{R_3} \tag{7.4}$$

The total current through the circuit is $I = \frac{V}{R_T}$, where R_T is the effective resistance in this circuit.

$$I = I_1 + I_2 + I_3 \tag{7.5}$$

$$\therefore \frac{V}{R_T} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \tag{7.6}$$

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{7.7}$$

Thus, in parallel combination, the reciprocal of the effective resistance is equal to the sum of the reciprocals of individual resistance. Therefore, the effective resistance in a parallel combination is smaller than the value of the smallest resistance.

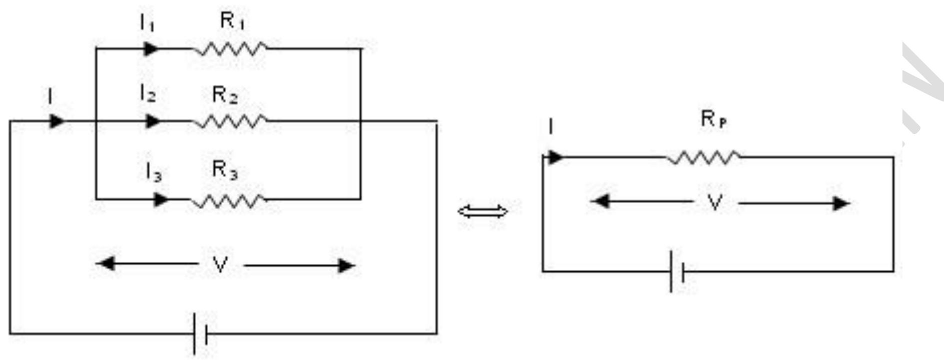


Figure 7.2

7.2 Kirchhoff's Rules to Closed Electrical Circuit

Current flow in circuits is produced when charge carriers travel through conductors. Current is defined as the rate at which this charge is carried through the circuit. A fundamental concept in Physics is that charge will always be conserved. In the contexts of circuits this means that, since current is the rate of flow of charge, the current flowing into a point must be the same as current flowing out of that point.

Robertz Kirchhoff's proposed two general rules for solving circuit network.

7.2.1 Kirchhoff's Current Law (KCL)

Kirchhoff's Current Law (Kirchhoff's First Rule) states that at any junction of several circuit elements, the sum of currents entering a junction is equal to the sum of currents leaving the junction. If the current entering a junction is taken as positive and current leaving the junction as negative, then Kirchhoff's Current Law can be stated as "The algebraic sum of current meeting at any junction in a circuit is zero".

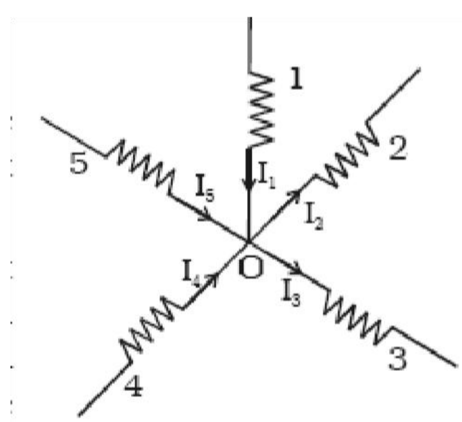


Figure 7.3 Kirchhoff's Current Law

Consider the diagram shown in Figure 7.3 in which 1,2,3,4 and 5 are the conductors meeting at a junction O and I_1 , I_2 , I_3 , I_4 and I_5 are the currents passing through the conductors respectively. According to Kirchhoff's Current Law, the currents I_1 , I_4 and I_5 are entering the junction while currents I_2 and I_3 are leaving the junction.

$$\therefore\ I_1 + (-I_2) + (-I_3) + I_4 + I_5 = 0 \qquad (7.8a)$$

Or

$$I_1 + I_4 + I_5 = I_2 + I_3 \qquad (7.8b)$$

Therefore, the sum of currents entering the junction is equal to the sum of the currents leaving the junction. This law is a consequence of conservation of charges.

7.2.2 Kirchhoff's Voltage Law (KVL)

Kirchhoff's Voltage Law (Kirchhoff's Second Rule) states that the algebraic sum of the products of resistances and currents in each part of any closed loop of electrical network is equal to the algebraic sum of the electromotive forces acting in that loop. This law is a consequence of conservation of energy for electrical circuits.

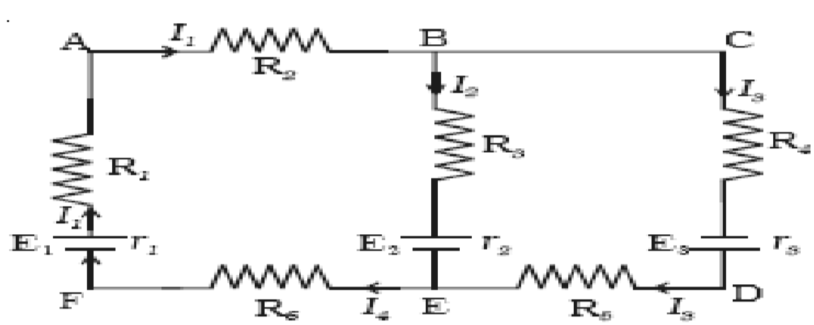


Figure 7.4 Kirchhoff's Voltage Law

In more general form, Kirchhoff's Voltage Law states that the algebraic sum of all the potential differences along a closed loop in a circuit is equal to zero.

In applying Kirchhoff's laws to electrical circuits, sign convention is very important. The direction of current flow may be assumed either clockwise or anticlockwise. The current in clockwise direction is taken as positive while the current in the anticlockwise direction is taken as negative. The electromotive force is taken as positive when we traverse from negative to positive electrode through the cell. Let us consider the electric circuit shown in Figure 7.4. If Kirchhoff's rule is applied to the closed loop ABCDEFA, we have:

$$I_1 R_2 + I_3 R_4 + I_3 r_3 + I_3 R_5 + I_4 R_6 + I_1 r_1 + I_1 R_1 = E_1 + E_3 \tag{7.9}$$

Both cells E_1 and E_3 send currents in clockwise direction. For the closed loop ABEFA,

$$I_1 R_2 + I_2 R_3 + I_2 r_2 + I_4 R_6 + I_1 r_1 + I_1 R_1 = E_1 - E_2 \tag{7.10}$$

Negative sign in E_2 indicates that it sends current in the anticlockwise direction.

7.3 Electrical Energy and Power

Electrical energy is defined as the capacity to do work.

When an electric current I, flows through a conductor of resistance R in time t, electrical energy is used in overcoming the resistance of the wire. Therefore, the quantity of charge flowing is $q = I\,t$. The work done W, or the energy expended in moving the charge q between two points having a potential difference V is:

$$W = qV = V\,I\,t \tag{7.11}$$

The electrical energy expended is converted into heat energy and this conversion is called the heating effect of electric current. The heat generated in Joules when a current of 1 ampere flows through a resistance of R ohm for t seconds is given by:

$$H = I^2 Rt = \frac{V^2 t}{R} \tag{7.12}$$

This relation given by equation (7.12) is known as Joule's law of electrical heating. The S.I unit of electrical energy is Joule. It can also be measured in kilowatt hour (kWh).

Then, electric power is defined as the time rate of doing electric work. Its S.I unit is Watt. Watt is practically defined as the rate at which work is being done in a conductor in which a current of 1 ampere is flowing when the voltage applied is 1 volt.

$$\therefore \quad power = \frac{work\ done}{Time} = \frac{V\,I\,t}{t} = I\,V \tag{7.13}$$

Electric power is the product of potential difference and current.

$$P = IV = I^2 R = \frac{V^2}{R} \tag{7.14}$$

Worked Examples

1. An electric cable contains a single copper wire of radius 9 mm and its resistance is 5Ω. This cable is replaced by six insulated copper wires, each of radius 3 mm. Determine the effective resistance of the cable.
   Solution:
   Initially the resistance of the cable is: $R = \frac{\rho l}{A} = \frac{\rho l}{\pi\left(9\times10^{-3}\right)^2} = 5\Omega$
   Resistance of each insulated copper wire is: $R' = \frac{\rho l}{A} = \frac{\rho l}{\pi\left(3\times10^{-3}\right)^2}$
   Equivalent resistance of cable, $R_{equi.} = \frac{R'}{6} = \frac{1}{6}\left(\frac{\rho l}{\pi\left(9\times10^{-6}\right)}\right) = \frac{1}{6}\times 9R$
   $$R_{equi.} = \frac{3}{2}\times 5\Omega = 7.5\,\Omega$$

2. Two uniform wire A and B are of the same metal and have equal masses. The radius of wire A is twice that of wire B. Determine the total resistance of wire A and B when connected in parallel.
   Solution:
   The density and masses of the wire are the same, therefore, their volumes are also equal.
   $$\frac{m}{\rho} = v \Rightarrow v_A = v_B \;\; or \;\; A_A l_A = A_B l_B$$
   $$\frac{R_A}{R_B} = \frac{\rho_A l_A}{A_A}\times\frac{A_B}{\rho_B l_B} = \frac{A_B}{A_A}\times\frac{l_A}{l_B} = \left(\frac{A_B}{A_A}\right)^2 = \left(\frac{r_B}{r_A}\right)^4$$
   But, $r_A = 2r_B \Rightarrow \frac{R_A}{R_B} = \frac{1}{16} \Rightarrow R_B = 16R_A$
   Resistance R_A and R_B are connected in parallel, their equivalent resistance, R is:
   $$R = \frac{R_A\,R_B}{R_A + R_B} = \frac{(R_A)(16R_A)}{(R_A + 16R_A)} = \frac{16}{17}R_A$$

3. In the circuit shown in figure7.5 below, determine the current in the circuit.

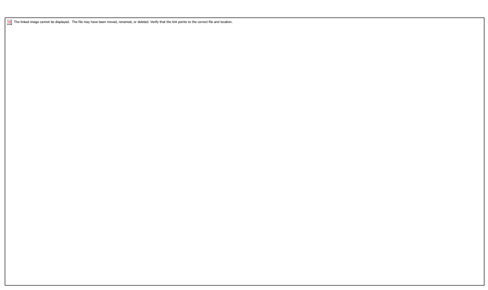


Figure 7.5
Solution:

The distribution of the current is shown in the circuit below:

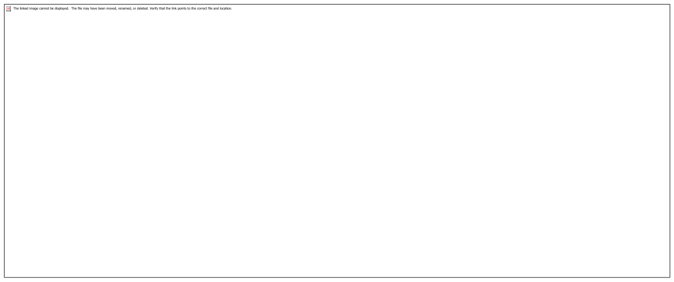


Figure 7.6

The elements are connected in series, therefore, the current in all of them will be the same.

Let the current be I. Applying Kirchhoff's voltage law in ABCDA loop gives:

$$10 + 4i - 20 + i + 15 + 2i - 30 + 3i = 0$$

$$\therefore\ 10i = 25 \quad \Rightarrow i = 2.5A$$

4. Consider the circuit shown in figure 7.7 below, determine the current in the circuit.

Figure 7.7

Solution:

The distribution of current in the circuit gives:

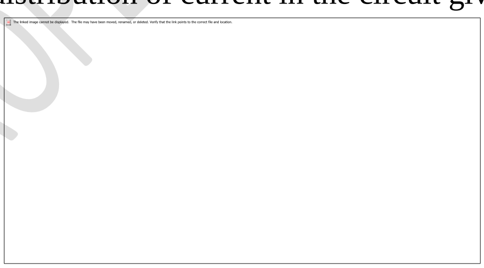


Figure 7.8

Applying Kirchhoff's voltage law in loop ABEFA gives:

$$i_1 + 30 + 2(i_1 + i_2) - 10 = 0$$

$$3i_1 + 2i_2 + 20 = 0 \qquad \text{(i)}$$

Applying Kirchhoff's voltage law in BCDEB gives:

$$+30 + 2(i_1 + i_2) + 50 + 2i_2 = 0$$

$$4i_2 + 2i_1 + 80 = 0 \qquad \text{(ii)}$$

$$2i_2 + i_1 + 40 = 0 \qquad \text{(iii)}$$

Solving equations (i) and (iii) simultaneously yields:

$$i_1 = 10A \ , \quad i_2 = -25A$$

Therefore, $i_1 - i_2 = -15A$

Current in wire AF = 10A from A to E

Current in wire EB = 15A from B to E

Current in wire DE = 25A from D to C

5. An electric iron of resistance 80Ω is operated at $200V$ for two hours. Calculate the electrical energy consumed.
   Solution:
   Electrical energy = Power x time
   $$P = \frac{V^2}{R} = \frac{(200)^2}{80} = 500W$$
   Electrical energy $= (500)(2) = 1\,Kwh$
6. A 220 V potential difference is maintained across an electric heater that is made from nichrome wire of resistance 20Ω.
   (a) Determine the current in the wire and the power rating of the heater.
   (b) At an estimated price of ₦ 2.00k per kilowatt-hour of electricity, what is the cost?
   Solution:
   (a) Using $\Delta V = IR \Rightarrow I = \frac{\Delta V}{R} = \frac{220V}{20\Omega} = 11A$
   $$P = I^2 R = (11A)^2 (20\Omega) = 2{,}420\,W = 2.42\,kW$$
   (b) The amount of energy transferred in time is:
   $$\Delta t = P\,\Delta t = (2.42\,kW)(2\,hr) = 4.84\,kWh$$
   If energy is purchased at ₦ 2.00k per kilowatt-hour, then the cost is:
   $$\text{Cost} = (4.84\,kWh)\left(\frac{2}{1\,kWh}\right) = \text{₦ } 9.68\text{k}$$

Self- Assessment Questions (SAQs)

1. The effective resistance when two resistors are connected in series and parallel are 10Ω and $2.4\,\Omega$ respectively. Determine the resistance of individual resistor.

2. Determine the resistance of the combined resistors shown in figure 7.9 below. The resistance of each resistor is R.

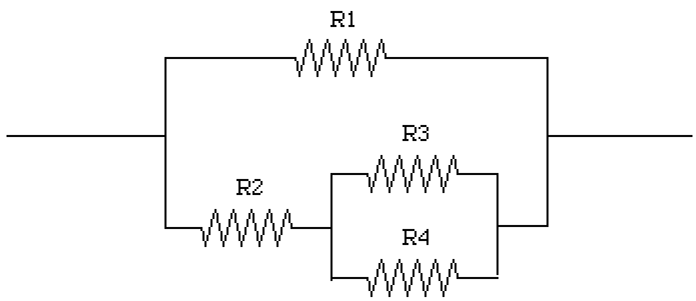


Figure 7.9

3. A heating element is maintained by a potential difference of $75.0\,V$ across the length of a nichrome wire that has a cross-sectional area of $2.60\times10^{-6}\ m^2$. Nichrome has a resistivity of $5.0\times10^{-7}\ \Omega m$.
   (a) If the element dissipates $500\,W$, determine its length.
   (b) If a potential difference of 100 V is used to obtain the same dissipation rate, what should the length be?

4. An unknown resistor is connected between the terminals of a 3.0 V battery. Energy is dissipated in the resistor at the rate of 0.540 W. The same resistor is then connected between the terminals of a 1.50 V battery. At what rate is energy now dissipated?

5. A wire with a resistance of 6.0Ω is drawn out through a die so that its new length is three times its original length. Calculate the resistance of the longer wire, assuming that the resistivity and density of the material are unchanged.

6. Consider the circuit shown in fig. 7.10, given that $E_1 = 6.0\,V$, $E_2 = 10.0\,V$ and $R_1 = 2.0\,\Omega$. Determine the value of the resistance R_2 if the current that passes through this resistance is 2.0 A.

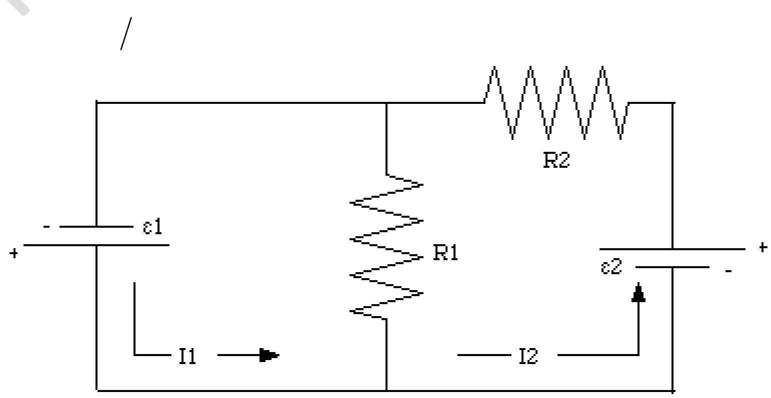


Figure 7.10

7. A certain x-ray tube operates at a certain current of 7.0 mA and a potential difference of 80 kV, determine its power.

SUMMARY

In this session, you have learnt that:

1. Two or more resistors are said to be connected in series when they are connected in such a way that the same current flows through all the resistors. The effective resistance of the series combination of a number of resistors is equal to the sum of the resistances of each resistor. The effective resistance of three resistors R_1 , R_2 and R_3 is:

$$R_T = R_1 + R_2 + R_3$$

2. A number of resistors are said to be connected in parallel when they are connected in such a way that the same potential difference is maintained across each of the resistor. In parallel combination, the reciprocal of the effective resistance is equal to the sum of the reciprocals of individual resistance. The effective resistance of three resistors R_1 , R_2 and R_3 is:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

3. Kirchhoff's Current Law states that the algebraic sum of current meeting at any junction in a circuit is zero.
4. Kirchhoff's Voltage Law states that the algebraic sum of all the potential differences along a closed loop in a circuit is equal to zero.
5. The heat generated in Joules when a current of 1 ampere flows through a resistance of R ohm for t seconds is given by:

$$H = I^2 Rt = \frac{V^2 t}{R}$$

6. The power consumed in an electrical circuit through Joule heating is given by:

$$P = IV = I^2 R = \frac{V^2}{R}$$

Study Session 8: The Wheatstone Bridge and Potentiometer and Their Applications

Expected Duration: 1 week or 2 contact hours

Introduction

Physics is an experimental science which unfold laws of nature that requires the use of instruments. In this session, you will learn about potentiometer which is a very versatile instrument. It can be used to measure resistance as well as electromotive force. The application of Wheatstone bridge will also be discussed.

Learning Outcomes

When you have studied this session, you should be able to explain the:

8.1 Wheatstone Bridge

8.2 Methods Applied in Wheatstone Bridge to Determine Unknown Resistance

8.3 Metre Bridge or Slide Wire Bridge

8.4 Potentiometer

8.1 Wheatstone Bridge

Wheatstone bridge was invented by Charles Wheatstone and it is used to measure the unknown resistance connected in electrical circuits. Wheatstone bridge is also known as a metre bridge or slide wire bridge. It consists of four resistors of which two resistors are known resistors, one variable resistor and one unknown resistor. It also consists of a galvanometer.

Consider the diagram of Wheatstone bridge shown in Figure 8.1. It consists of four arms PQ, QR, RS and PS which has fixed and variable resistors. R_1 and R_2 are the fixed resistors, R_3 is the variable resistor and R_x is the unknown resistor. The variable resistor restricts and control the flow of electric current. The arms PQ and QR are known as **Ratio Arms.** A galvanometer is connected between the terminals Q and S. Q and S is called the galvanometer arm. A battery is connected to the other two terminals P and R. P and R is called the **Battery Arm.** By adjusting the value of variable resistor, the deflection in galvanometer can be made as null.

Figure 8.1 Wheatstone bridge

8.2 Methods applied in Wheatstone bridge to determine unknown resistance

There are three methods that can be employed by Wheatstone bridge to determine the unknown resistance. They include:

8.2.1 **Method 1**

The bridge is said to be balanced when there is no current flowing through the galvanometer. This means that the potential difference between the points Q and S is zero. In this case, the current flowing through the fixed resistors R_1 and R_2 is the same and is considered as I_1. The current flowing through the variable resistor R_3 and the unknown resistor R_x will be the same and is I_2. As the potential at Q and S is the same, the voltage drop from the point P to Q is equal to the voltage drop from point P to point S. Therefore,

$$I_1 R_1 = I_2 R_3 \tag{8.1}$$

Now, the voltage drop from point Q to point R is equal to the voltage drop from point S to R. Thus:

$$I_1 R_2 = I_2 R_x \tag{8.2}$$

Dividing equation (1) by equation (2) gives:

$$\frac{R_1}{R_2} = \frac{R_3}{R_x} \tag{8.3}$$

$$R_x = \frac{R_3\, R_2}{R_1} \tag{8.4}$$

The unknown resistance R_x can be determined in terms of other known resistors in the bridge.

8.2.2 : Method 2

The value of the unknown resistance can also be determined by applying Kirchhoff's rule in the same Figure 8.1 above. When the current passing through the galvanometer, I_g is zero, the bridge is said to be balanced. Kirchhoff's rule is now applied to the closed loops PSQP and RQSR. Applying the Kirchhoff's Voltage Law, the algebraic sum of all voltages across the circuit will be zero. Considering the first loop PSQP,

$$-I_2\, R_3 + 0 + I_1\, R_1 = 0 \tag{8.5}$$

Similarly, loop RQSR gives:

$$I_1\, R_2 + 0 - I_2\, R_x = 0 \tag{8.6}$$

The first loop gives:

$$\frac{I_2}{I_1} = \frac{R_1}{R_3} \tag{8.7}$$

Also, second loop gives:

$$\frac{I_2}{I_1} = \frac{R_2}{R_x} \tag{8.8}$$

Thus, $\frac{R_1}{R_3} = \frac{R_2}{R_x}$ (8.9)

$$\therefore\ R_x = \frac{R_2\, R_3}{R_1} \tag{8.10}$$

8.2.3 Method 3

In this method, Ohm's law is applied to the same figure 8.1 to determine the value of the unknown resistance. By adjusting the variable resistor, the bridge will be balanced. This implies that, the voltage at points Q and S are equal. Recall, from Ohm's law:

$$V = I R \quad \Rightarrow I = \frac{V}{R} \tag{8.11}$$

$$\therefore \; I_2 = \frac{V}{R_x + R_3} \tag{8.12}$$

$$\text{The voltage at point S} = V\left(\frac{R_x}{R_x + R_3}\right) \tag{8.13}$$

$$\text{The voltage at point Q} = V\left(\frac{R_2}{R_1 + R_2}\right) \tag{8.14}$$

$$\text{Thus, } V_{SQ} = V\left(\frac{R_x}{R_x + R_3}\right) - V\left(\frac{R_2}{R_1 + R_2}\right) \tag{8.15}$$

$$\text{When } V_{SQ} = 0 \quad \Rightarrow V\left(\frac{R_x}{R_x + R_3}\right) = V\left(\frac{R_2}{R_1 + R_2}\right) \tag{8.16}$$

$$\text{This gives: } R_1 R_x + R_2 R_x = R_2 R_3 + R_2 R_x \tag{8.17}$$

$$R_1 R_x = R_2 R_3 \tag{8.18}$$

$$\therefore \; R_x = \frac{R_2 R_3}{R_1} \tag{8.19}$$

8.2.4 Applications of Wheatstone Bridge

With the help of Wheatstone bridge, we can have a light detector circuit. Bridge circuits are used to measure intensity of light. In the bridge circuit, one of the resistor is replaced by light dependent resistor. Thus, the deviation in the levels of light can be monitored and measured. The bridge circuit can be used to measure changes in pressure in strain gauge, thermistor and potentiometer. Wheatstone bridge is used with operational amplifier to measure and amplify small changes that takes place in resistors. The bridge circuit is used in transducers and sensors in industries.

One important application of the Wheatstone bridge includes the strain gauge. A strain gauge is used to measure strain. Strain is the amount of change a material undergoes when an external force is applied. When a metal conductor is stretched or compressed, the resistance of the material changes as the length, diameter and the resistivity of the material changes. The resistance changes to a very small value for a particular strain. Wheatstone bridge is used to measure accurately the small resistance changes. In this case, the unknown resistance is replaced with a strain gauge.

8.3 Metre Bridge or Slide Wire Bridge

Metre-bridge is a sensitive device used to determine the resistance of a conductor (wire). It is based on the principle of Wheatstone –bridge.

Figure 8.2: Metre- Bridge

A metre bridge as shown in figure 8.2 consists of one metre long wire of manganin or constantan which is fixed along a scale on a wooden base AC.The area of cross-section of the wire is the same at all places. The ends A and C of the wire are joined to two L-shaped copper strips carrying binding-screws. In between these strips, leaving a gap on either side, there is a third copper strip having three binding screws. The middle screw D is connected to a sliding jockey B through a stunted-galvanometer G. The knob of the jockey can be made to touch any point on the wire.

The connection as shown in figure 8.2 can be used to measure an unknown resistance. A resistance R is taken out from the resistance box and the key K is closed. Now the jockey is slided along the wire and a point is determined such that, on pressing the jockey on the wire at that point there is no deflection in the galvanometer G. In this position, the points B and D are at the same potential. The point B is called 'null-point'. The lengths of both parts AB and BC of the wire are measured. Suppose the resistance of the length AB of the wire is P and that of length BC is Q. Then, by the principle of Wheatstone-bridge, gives:

$$\frac{P}{Q} = \frac{R}{S} \qquad (8.20)$$

Let the length AB be l cm. Then the length BC will be ($100 - l$) cm.

Therefore, resistance of AB, that is, $P = \frac{\rho l}{A}$, and resistance of BC, Q $= r\frac{(100 - l)}{A}$

Where r is the specific resistance of the material of the wire and 'A' is the area of cross-section of the wire. Thus,

$$\frac{P}{Q} = \frac{l}{(100 - l)} \qquad (8.21)$$

By substituting equation (8.20) into equation (8.21) gives:

$$\frac{l}{(100 - l)} = \frac{R}{S} \quad \Rightarrow S = R\frac{(100 - l)}{l} \qquad (8.22)$$

R is the resistance taken in the resistance box and *l* is the measured length. Hence, the value of resistance S can be determined from the above equation.

8.4 Potentiometer

A potentiometer is a device used for measuring the potential difference between two points. It can also be used to compare the electromotive forces of two cells or to measure internal resistance of a cell. It consists of a number of segments of constantan or manganin wire of uniform area of cross section stretched on a wooden board between two thick copper strips. Each segment of wire is 100 cm long. A metre rod is fixed parallel to its length. A battery connected across the two ends terminals provides current through the wire which is kept constant by using a rheostat.

Figure 8.3 Potentiometer

8.4.1 Principle of Potentiometer

The potentiometer is based on the principle that when a constant current is passed through a wire of uniform area of cross-section, the potential drop across any portion of the wire is directly proportional to the length of that portion.

Let V be potential difference across certain portion of wire whose resistance is R. If I is the current passing through the wire, then V = I R

But $R = \frac{\rho l}{A}$ (8.23)

Where *l*, A and ρ are length, area of cross-section and resistivity of the material of wire respectively.

Therefore, $V = \frac{I \rho l}{A}$ (8.24)

If a constant current passed through the wire of uniform area of cross-section, then, I and A are constants. Since, for a given wire, ρ is also constant. We have

V = constant x l (8.25)

Therefore, V is directly proportional to *l*

Hence, if a constant current flows through a wire of uniform area of cross-section, then potential drop along the wire is directly proportional to the length of the wire.

A potentiometer is said to be more sensitive if it measures a small potential difference more accurately. In order to increase the sensitivity of a potentiometer, the length of potentiometer wire will have to be increased so that the length may be measured more accurately.

8.4.2 Comparison of Electromotive Forces of Two Cells Using Potentiometer

Two cells, whose electromotive forces (emfs) are E_1 and E_2 can be compared by making use of the circuit shown in figure 8.4. The positive poles of both cells are connected to the terminal A of the potentiometer. The negative poles of the two terminals are connected to terminals 1 and 2 of a two way key while its common terminal is connected to a jockey j through a galvanometer G. An auxiliary or driver battery of emf E_ϕ, an ammeter A, rheostat Rh and a one way key K are connected between the end terminals A and B of the potentiometer. Thus, the positive poles of the two cells as well as the positive pole of auxiliary battery are connected at the common point A. It is important to note that the e.m.f of auxiliary battery is always greater than the e.m.f of either of the two cells.

Figure 8.4 Comparison of emf of two cells

To compare the e.m.fs of the two cells, a constant current is passed through the potentiometer wire between points A and B. The current is kept constant by using the rheostat. When the plug is put in the gap between the terminals 1 and 3 of the two way key, the cell of emf E_1 is connected in the circuit. Suppose the balancing length (between A and J) is I_1. If x is the resistance per unit length of the potentiometer wire and I is the constant current flowing through it, then:

$$E_1 = (x l_1) I \tag{8.26}$$

When the key is connected in the gap between the terminals 2 and 3 and removed from the gap between 1 and 3, the cell of emf E_2 is connected in the circuit. Let the balancing length be I_2 in this case. Then,

$$E_2 = (x l_2) I \tag{8.27}$$

Dividing equation (8.26) by equation (8.27) gives:

$$\frac{E_1}{E_2} = \frac{l_1}{l_2} \tag{8.28}$$

8.4.3 Measurement of Internal Resistance of a Cell using Potentiometer

The internal resistance of a cell can be determined by using a potentiometer as shown in figure 8.5 below. A constant current I is maintained through the potentiometer wire by using rheostat. The key K_2 is opened and the jockey is moved over the potentiometer wire so as to balance the e.m.f of the cell whose internal resistance is to be determined. Let l_1 be the balancing length of the potentiometer wire between A and jockey j. If x is the resistance per unit length of the wire, then,

$$E = (x\, l_1) I \qquad (8.29)$$

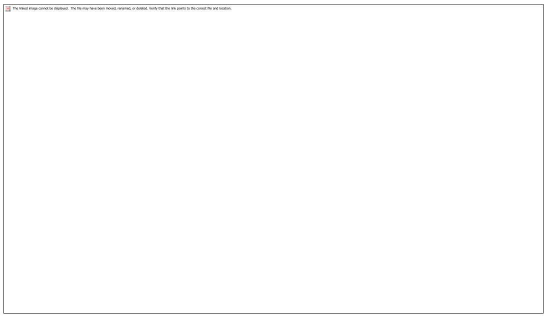


Figure 8.5 Measurement of internal resistance of a cell

A resistance S is introduced from a resistance box S and the key K_2 is closed. The balance point for the terminal potential difference V between the two terminals of the cell is determined. If I_2 is the balancing length, then:

$$V = (x l_2) I \qquad (8.30)$$

By dividing equation (8.29) by equation (8.30) gives:

$$\frac{E}{V} = \frac{l_1}{l_2} \qquad (8.31)$$

The internal resistance of the cell is given by:

$$r = \left(\frac{E}{V} - 1\right) S \qquad (8.32)$$

Substituting equation (8.31) into equation (8.32) gives:

$$r = \left(\frac{l_1}{l_2} - 1\right) S \qquad (8.33)$$

If the values of l_1 , l_2 and S are known, then the internal resistance r of cell can be determined.

Worked Examples

1. A Daniel cell is balanced on 125 cm length of a potentiometer wire. When the cell is short circuited with a 2Ω resistance, the balancing length obtained is 100 cm. Calculate the internal resistance of the cell.

   Solution:

Using $r = \left(\frac{l_1}{l_2} - 1\right)S \quad \Rightarrow r = \left(\frac{l_1 - l_2}{l_2}\right)S = \left(\frac{125 - 100}{100}\right)\times 2$

$r = 0.5\,\Omega$

2. The resistivity of a potentiometer wire is $40 \times 10^{-8}\ \Omega m$ and its area of cross-section is $8 \times 10^{-6}\ m^2$. If the current flowing through the wire is 0.2 A, determine the potential gradient.

   Solution:

   Potential gradient $= \frac{V}{L} = \frac{i\,R}{L} = \frac{i\,\rho L}{AL} = \frac{i\,\rho}{A}$

   $$= \frac{0.2 \times 40 \times 10^{-8}}{8 \times 10^{-6}} = 10^{-2}\ V\,m^{-1}$$

3. In an experiment to measure the internal resistance of a cell by potentiometer, it is found that the balance point is at a length of 2 m when the cell is shunted by a 5Ω resistance; and is at a length of 3 m when the cell is shunted by a $10\,\Omega$ resistance. Calculate the internal resistance of the cell.

   Solution:

   Using $r = \left(\frac{l_1 - l_2}{l_2}\right)S \quad \Rightarrow r = \left(\frac{l_1 - 2}{2}\right)\times 5$ (i)

   And $r = \left(\frac{l_1 - 3}{3}\right)\times 10$ (ii)

   Solving equation (i) and (ii) gives: $r = 10\,\Omega$

4. In the circuit diagram shown below, calculate the current through the galvanometer across P and R of $10\,\Omega$ resistances with a potential difference of 20 V.

Figure 8.6

Solution:

Considering loop PRQP:

$$50\, i_1 - 30\, i_2 + 10\, i_g = 0 \quad \Rightarrow 5 i_1 - 3 i_2 + 1 i_g = 0 \qquad (1)$$

Consider the loop PSQP: $100(i_1 - i_g) - 40(i_2 + i_g) - 10 i_g = 0$ (2)

Consider the loop RQSVR: $30 i_2 + 40(i_2 + i_g) = 20$ (3)

Solving equations (1) (2) and (3) yields $i_g = 0.0315\, A$

Therefore, the current that flows across the galvanometer i_g is 0.0315 A

5. In the circuit diagram below, calculate the voltage across the point P and R. Also, determine the resistance R_4 required to balance the bridge circuit.

Figure 8.7

Solution:

Consider the arm $P_1 Q_1 P_2$: $V_{Q1} = \dfrac{R_3 V_5}{(R_1 + R_3)} = \dfrac{40 \times 100}{90} = 44V$

Consider the next arm $P_1 Q_2 P_2$: $V_{Q2} = \dfrac{R_4 V_5}{(R_2 + R_4)} = \dfrac{50 \times 100}{150} = 33V$

Thus, $V_{out} = 44 - 33 = 11V$

$$R_4 = \frac{R_2 R_3}{R_1} = \frac{40 \times 100}{50} = 80\ \Omega$$

Self – Assessment Questions (SAQs)

1. In a Wheatstone's bridge, if the galvanometer shows zero deflection, determine the unknown resistance. Given that, $P = 1000\ \Omega$ $\quad Q = 10{,}000\ \Omega$ and $R = 20\ \Omega$.

2. In a metre bridge, the balancing length for a $10\,\Omega$ resistance in left gap is 51.8 cm. Find the unknown resistance and specific resistance of a wire of length 108 cm and radius 0.2 mm.
3. In a potentiometer circuit, balance point is obtained at 45 cm from end A when an unknown e.m.f is measured. The balance point shifts to 30 cm from this end when a cell of 1.02 V is put in the circuit. Standard cell E always supplies a constant current. Calculate the value of unknown emf.
4. A potentiometer circuit is used to compare the e.m.f of two cells E_1 and E_2. The balance point is obtained at lengths 30 cm and 45 cm, respectively for E_1 and E_2. Calculate the e.m.f of E_1 if E_2 is 3.0 V.

SUMMARY

In this session, you have learnt that:

1. The Wheatstone bridge circuit is used to measure accurately an unknown resistance, R_x by comparing it with known resistances (R_1, R_2 and R_3).

$$R_x = \frac{R_3\,R_2}{R_1}$$

2. Wheatstone bridge is used with operational amplifier to measure and amplify small changes that takes place in resistors. The bridge circuit is used in transducers and sensors in industries.
3. Metre-bridge is a sensitive device used to determine the resistance of a conductor (wire). It is based on the principle of Wheatstone –bridge.
4. A potentiometer is a device used for measuring the potential difference between two points. It can also be used to compare the electromotive forces of two cells or to measure internal resistance of a cell
5. The potentiometer is based on the principle that when a constant current is passed through a wire of uniform area of cross-section, the potential drop across any portion of the wire is directly proportional to the length of that portion.

Study Session 9: Electrodynamics of Charged Particles

Expected Duration: 1 week or 2 contact hours

Introduction

Both Electricity and Magnetism have been known for more than 2000 years. However, it was discovered by Oersted in 1820, that they were intimately related. Now, it is known that all magnetic phenomena result from forces arising from electric charges in motion. When a charged particle moves in a magnetic field, it is acted on by the magnetic force and the motion is determined by Newton's law. In this session, you will learn how magnetic field exerts forces on moving charged particles like electrons and protons. You will also learn how particles can be accelerated to very high energies in a cyclotron. Velocity selector, Mass Spectrometer and Hall Effect will also be discussed under the applications of charged particles in an electric and magnetic fields.

Learning Outcomes

When you have studied this session, you should be able to explain:

9.1 The Magnetic Force on a Moving Charge

9.2 The Motion of a Charged Particle in a Uniform Magnetic Fields

9.3 The Motion of a Charged Particle Projected at an Angle into a Magnetic Field

9.4 Charged Particles in an Electric and Magnetic Fields

9.1 The Magnetic Force on a Moving Charge

A magnetic field exists at a particular point in space if a force is exerted on a moving charge at that point. A moving charge or current creates a magnetic field in the surrounding space in addition to its electric field. The field by which magnetic interaction occurs is also called magnetic field denoted by the symbol $\vec{B}$. The S.I unit for $\vec{B}$ is Tesla or Webers/m^2 . In CGS, the unit is Gauss (G). 1T = 10^4 G. The magnetic field of the earth is of the order of 10^{-4} T or 1 G. The magnetic force acting on a charge q moving with velocity v is given by:

$$\vec{F} = q\left(\vec{V} \times \vec{B}\right) \tag{9.1}$$

In vector form, $\vec{F}$ can be written as:

$$\vec{F} = q\vec{V}\times\vec{B} = q\begin{vmatrix} i & j & k \\ V_x & V_y & V_z \\ B_x & B_y & B_z \end{vmatrix} \tag{9.2}$$

Therefore, the magnitude of the magnetic force on charge q is given by:

$$F = |q|\, V\, B \sin\theta \tag{9.3}$$

Where $\theta =$ angle measured from the direction of $\vec{V}$ to the direction of $\vec{B}$.

$|q|$ = magnitude of the charge.

The direction of $\vec{V}\times\vec{B}$ and the direction of $\vec{F}$ for both positive and negative charge q can be determined from the right-hand rule. $\vec{F}$ has the direction of $\vec{V}\times\vec{B}$ if q is positive while $\vec{F}$ has the direction of $-\vec{V}\times\vec{B}$ if q is negative . Similarly, equation (9.2) indicates that:

$F = 0$ (when $\vec{V}$ is parallel to $\vec{B}$ and when $\vec{V}$ = 0)

$F_{\max} = qV\,B$ (when $\vec{V}$ is perpendicular to $\vec{B}$)

9.2 Motion of a Charged Particle in a Uniform Magnetic Field

A static magnetic field does not exert any force on a charged particle at rest. It can experience a magnetic force only when it enters the magnetic field with velocity v. The force is given by $F = qV\,B\sin\theta$.This force is always perpendicular to the direction of motion of the charged particle. It will only change the direction of motion but not the magnitude of the velocity. Let us consider the motion of a charged particle in a uniform magnetic field with strength B that is perpendicular to the velocity v, then the magnitude of the magnetic force is given by:

$$F = qV\,B \tag{9.4}$$

And its direction is perpendicular to V. When the initial velocity of a positively charged particle is perpendicular to the magnetic field, the particle's orbit is a circle. The magnetic force is always directed towards the centre of a circular path, therefore, the magnetic force causes a centripetal acceleration. Whenever a particle moves in a circular path, it experiences a centripetal force. The magnitude of the magnetic force is equal to the product of the mass and centripetal acceleration. That is,

$$F = qV\,B = m\frac{V^2}{r} \tag{9.5}$$

Solving for r, gives:

$$r = \frac{mV}{qB} = \frac{P}{qB} \tag{9.6}$$

That is, the radius of curvature is proportional to the magnitude of the momentum mV of the charged particle and inversely proportional to the magnitude of the charge and to the magnitude of the magnetic field. Therefore, the radius depends on the mass to charge ratio of the charged particle. The distance travelled by the charged particle in one revolution is given by:

$$d = 2\pi r = 2\pi\left(\frac{mv}{qB}\right) \tag{9.7}$$

The angular frequency ω is the time it takes the charged particle to complete an orbit. The angular frequency of a charged particle in a constant magnetic field is given by:

$$\omega = \frac{V}{r} = \frac{qB}{m} \tag{9.8}$$

The period of motion of a charged particle in a constant magnetic field is given by:

$$T = \frac{2\pi r}{V} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \tag{9.9}$$

In other words, the charged particle undergoes oscillatory motion with a period proportional to the mass to charge ratio (m/q) and inversely proportional to the magnetic field. The frequency $f = \frac{1}{T} = \frac{qB}{2\pi m}$ (9.10)

is called the **cyclotron frequency.** Equation (9.9) shows that the cyclotron frequency is independent of the energy of the charged particle but depends only on its mass *m* and charge *q*.

These equations show that T , f and ω are independent of the speed V of the particle and the radius r of the orbit. If the velocity of the charged particle has two components, one perpendicular $(V\perp)$ to the uniform magnetic field and the other parallel $(V\|)$ to it, then the particle will move in a helical path about the direction of the magnetic field $\vec{B}$.

9.3 Motion of Charged Particle Projected at an Angle into a Magnetic Field

An electron of mass *m* and charge *q* with a uniform velocity V enters a uniform magnetic field B at an angle θ. It is assumed that the magnetic field is in z-direction. Thus, the electron's velocity can be resolved into a rectangular components V_x and V_z. The component of velocity parallel to the magnetic field is $V_z = V\cos\theta$ is not influenced by the field since

$$F_z = q(V_z \times B) = 0 \tag{9.11}$$

Hence, the electron continues to travel along the field lines with a velocity $V_z = V\cos\theta$.

The velocity component $V_x = V\sin\theta$ gives rise to a force on the electron.

$$F_x = q(V_x \times B) = qVB\sin\theta \tag{9.12}$$

Under the action of this force, the electron tends to describe a circular path in a plane perpendicular to the magnetic field. The radius of this path is given by:

$$R = \frac{mV_x}{qB} = \frac{mV\sin\theta}{qB} \tag{9.13}$$

The period of one revolution is given by:

$$T = \frac{2\pi R}{V_x} = 2\pi\left(\frac{mV\sin\theta}{qB}\right)\left(\frac{1}{V\sin\theta}\right) = \frac{2\pi m}{qB} \tag{9.14}$$

The resultant motion of the electron is obtained by superposition of the uniform translational motion parallel to B and the uniform circular motion in a plane normal to B. The resultant motion is along a helical path withaxis of the helix being the field direction. The pitch of the helix is the distance covered by the electron along the field direction in one revolution. Thus, the path pitch is given by:

$$P = V_z T = (V\cos\theta)T = V\cos\theta\left(\frac{2\pi m}{qB}\right) = \frac{2\pi mV\cos\theta}{qB} \tag{9.15}$$

9.4 Charged Particles in an Electric and Magnetic Fields

A charged particle moving in a region with an electric field $\vec{E}$ and magnetic field $\vec{B}$ will experience a total force $\vec{F}$ given by:

$$\vec{F} = q\vec{E} + q\vec{V}\times\vec{B} \tag{9.16}$$

This force is called the **Lorentz force**.

9.4.1 The Velocity Selector

A Velocity Selector is an electro-optic device which uses uniform electric and magnetic fields in cross-field configuration (perpendicular to each other) for setting a stream of charged particles of single velocity from a beam of particles having a wide range of velocities.

The electric field deflects the positively charged particles upward while the magnetic field deflects them downward as shown in Figure 9.1. If the magnitude of the electric field E and magnetic field B are adjusted such that the net force exerted on the electrons becomes zero, then,

$$F_E = F_M \quad \Rightarrow \quad qE = qBV \tag{9.17}$$

$$\therefore\ V = \frac{E}{B} \tag{9.18}$$

and the electrons will continue moving in a horizontal straight line through the region of the fields.

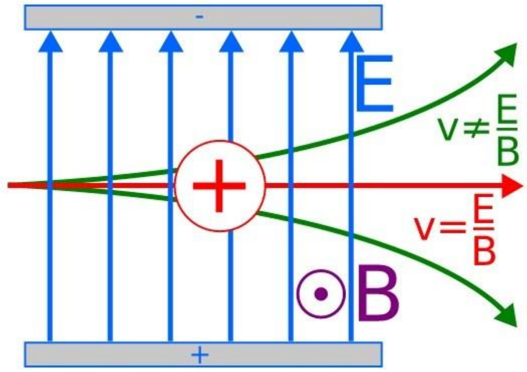


Figure 9.1: Charged particle moving in both Electric and Magnetic fields.

9.4.2 The Mass Spectrometer

A mass spectrometer is an instrument used to measure the mass or the mass-to-charge ratio for charged particles or ions.

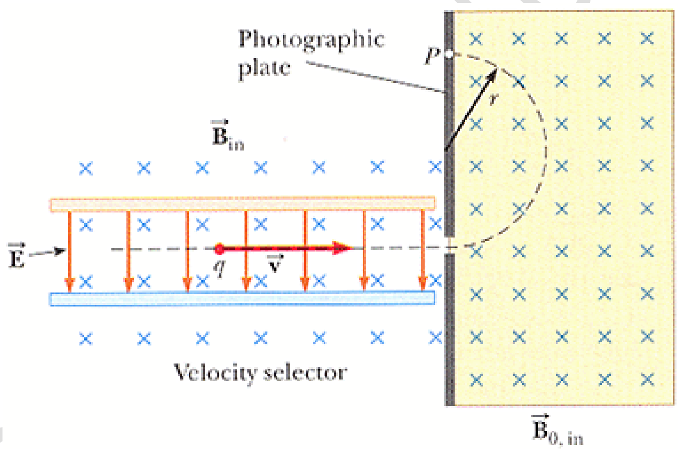


Figure 9.2: A mass spectrometer

The mass spectrometer shown in Figure 9.2 has source of charged particles where they are accelerated through a potential difference V. These particles pass through a slit into the velocity selector. Particles that have a speed of $v = \dfrac{E}{B}$ pass through the slit and enter a deflecting chamber of uniform magnetic field B. In this region, the particles move in a circular path of radius *r*. From equation (9.6), the mass can be expressed as follows:

$$m = \frac{r\,q\,B}{v} \qquad (9.19)$$

Using equation (9.18) , the mass-to-charge ratio can be expressed as:

$$\frac{m}{q} = \frac{B^2\,r}{E} \qquad (9.20)$$

If the charge *q* is known, then the mass *m* of the charged particle can be calculated in terms of B, E and r.

9.4.3 The Hall Effect

In 1879, Edwin Hall demonstrated that when a current I passes through a strip of metal which is placed perpendicular to a magnetic field $\vec{B}$, a potential difference is established in a direction perpendicular to both I and $\vec{B}$. This phenomenon is known as Hall effect. It is a technique used to determine the density and sign of charge carriers in a metal based on the forces exerted by crossed $\vec{E}$ and $\vec{B}$ fields on the charge carriers.

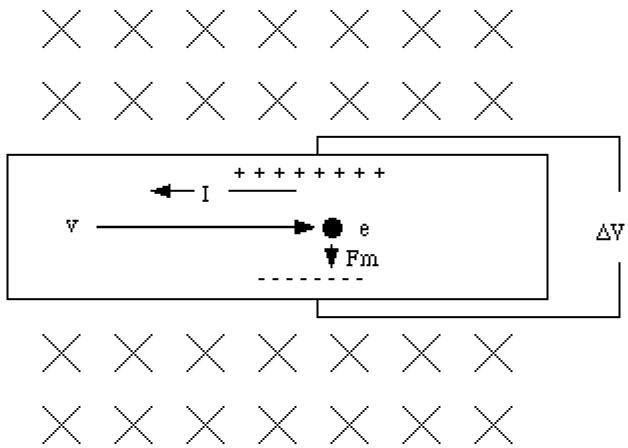


Figure 9.3 Current in a magnetic field

The diagram shown in Figure 9.3 consists of a metallic strip carrying a current in the direction shown and placed in a uniform magnetic field which is perpendicular to the electric field (which generates the current I). Suppose the charge carriers in the materials are electrons, then, the electrons will move in a direction opposite to that of the current. The magnetic field is perpendicular to the electric field and also perpendicular to the direction of motions of the electrons. As a result of the magnetic force the electrons are deflected downwards and an excess negative charge will be created on the bottom of the strip. At the same time, a deficit of negative charge will be created at the top of the strip. This charge distribution will generate an electric field that is perpendicular to the external electric field. At the equilibrium, the electric force produced by this field will cancel the magnetic force acting on the electrons. Equating the electric and magnetic forces gives:

$$\vec{F}_E = \vec{F}_B \Rightarrow eE_H = eV_d B \qquad (9.21)$$

when d is the width of the strip, the potential difference ΔV_H, called the Hall voltage across the strip is related to the electric field E_H by:

$$\Delta V_H = E_H d \qquad (9.22)$$

From equation (9.22), the drift speed V_d is related to the current I by:

$$I = neV_d A \tag{9.23}$$

where $A = t\,d$ is the cross-sectional area of the strip. Substituting E_H from equation (9.22) and V_d from equation (9.23) into equation (9.21) gives:

$$\Delta V_H = \frac{I\,B}{net} \tag{9.24}$$

Therefore, $\Delta V_H = R_H \dfrac{I\,B}{t}$ (9.25)

where $R_H = \dfrac{1}{ne}$ is called the Hall Coefficient. Equation (9.25) can be used to measure the magnitude of the magnetic fields and give information about the sign of the charge carriers and density.

Worked Examples

1. An electron moves through a uniform magnetic field given by $\vec{B} = B_x i + (3B_x)j$. At a particular instant, the electron has velocity $\vec{V} = (2.0i + 4.0\,j)ms^{-1}$ and the magnetic force acting on it is $\vec{F} = (6.4 \times 10^{-19}\ N)k$. Determine the magnitude of B_x.
   Solution:
   $\vec{F} = (-e)(\vec{V} \times \vec{B}) = (-1.6 \times 10^{-19})(2i + 4\,j) \times (B_x i + 3B_x\,j) = (6.4 \times 10^{-19}\,N)k$
   Simplifying the cross product gives: $2B_x = -4 \Rightarrow B_x = -2T$

2. A proton moves at a constant velocity of $50\ ms^{-1}$ along an axis through crossed electric and magnetic fields. Determine the electric field if the magnetic field is $B = (2.0\ mT)$.
   Solution:
   Since the velocity is constant, $F = 0 \Rightarrow qE + q\vec{V} \times \vec{B} = 0$
   $$|V| = \frac{|E|}{|B|} \Rightarrow |E| = 50\,ms^{-1} \times 2 \times 10^{-3}\ T$$
   $$|E| = 0.1\,Vm^{-1}$$

3. In an experiment with cosmic rays, a vertical beam of particles that have a magnitude 3e and a mass 12 times the proton mass enters a uniform horizontal magnetic field of 0.250 T. It bends in a semi- circle of 95 cm. Calculate the speed of the particle.
   Solution:
   Using $r = \dfrac{mv}{qB} \Rightarrow v = \dfrac{qBr}{m} = \dfrac{(3 \times 1.6 \times 10^{-19})(0.25)(0.475)}{(12 \times 1.67 \times 10^{-27})} = 2.84 \times 10^{6}\ ms^{-1}$

4. A proton travels with a speed of $3.0\times10^{6}\ ms^{-1}$ at an angle of 37^{0} with the direction of a magnetic field of 0.30 T in the positive y-direction. Determine the magnitude of the:
   (a) Magnetic force (b) proton's acceleration
   Solution:
   (a) $F = BqV\sin\theta = (0.30)(1.60\times10^{-19})(3.0\times10^{6})(\sin 37^{0}) = 8.7\times10^{-14}\,N$
   (b) $a = \dfrac{F}{m} = \dfrac{(8.7\times10^{-14})}{1.67\times10^{-27}} = 5.2\times10^{13}\ ms^{-2}$
5. A proton moves perpendicularly to a uniform magnetic field B at a velocity of $1.0\times10^{7}\ ms^{-1}$. If it experiences an acceleration of $2.0\times10^{13}\ ms^{-2}$, determine the magnitude of the magnetic field.
   Solution:
   $$B = \frac{F}{qV} = \frac{ma}{qV} = \frac{(1.67\times10^{-27})(2.0\times10^{13})}{(1.60\times10^{-19})(1.0\times10^{7})} = 0.20T$$
6. In a mass spectrometer, the electric field between the plates of the velocity selector is $E = 950\,V\,m^{-1}$ and the magnetic field B in both the velocity selector and in the deflection chamber has a magnitude of 0.9 T. Calculate the radius r for a singly charged ion of mass $m = 2.18\times10^{-26}\,kg$ in the deflection chamber.
   Solution:
   Using the fact that for ion to pass through the velocity selector undeflected, the force due to the electric and magnetic fields must balance. Therefore,
   $$qE = qVB \Rightarrow V = \frac{E}{B}$$
   When the ion enters the deflection chamber, it experiences a magnetic force, causing it to go in a circular orbit. The magnetic force then give rise to the centripetal acceleration as:
   $$F = qVB = m\frac{V^{2}}{r} \Rightarrow qB = m\frac{V}{r}$$
   $$r = \frac{mV}{qB} = \frac{mE}{qB^{2}} = \frac{(2.18\times10^{-26})(950)}{(1.6\times10^{-19})(0.9)^{2}} = 1.6\times10^{-4}\ m$$
   Thus, the radius of the orbit is 0.16 mm

Self – Assessment Questions (SAQs)

1. An electron emitted by a heated cathode and accelerated through a potential difference of 2.0 kV, enters a region with a uniform magnetic field of 0.15 T. Determine the trajectory of the electron if the field:
   (a) Is transverse to its initial velocity
   (b) Makes an angle of 30^{0} with the initial velocity

2. In a chamber, a uniform magnetic field of $6.5G\ (1G = 10^{-4}\,T)$ is maintained. An electron is shot into the field with a speed of $4.8\times10^{6}\ ms^{-1}$ normal to the field. Determine the radius of the circular path.
3. An electron is accelerated from rest by a potential difference of 350 V. It then enters a uniform magnetic field of magnitude 200 mT with its velocity perpendicular to the field. Calculate:
   (a) The speed of the electron
   (b) The radius of its path in the magnetic field.
4. An electron is accelerated through a potential difference of 1.0 kV and directed into a region between two parallel plates separated by 20 mm with a potential difference of 100 V between them. The electron is moving perpendicular to the electric field of the plates when it enters the region between the plates. Determine the uniform magnetic field applied perpendicular to both the electron path and the electric field that will allow the electron to travel in a straight line.
5. An electron that has velocity $V = (2.0\times10^{6}\ ms^{-1})i + (3.0\times10^{6}\ ms^{-1})j$ moves through the magnetic field $B = (0.030T)i - (0.15T)j$.
   (a) Determine the force on the electron
   (b) Repeat your calculation for a proton having the same velocity
6. An ion source in a mass spectrometer produces doubly ionized gold ions, each with a mass of $3.27\times10^{-25}\ kg$. The ions are accelerated from rest through a potential difference of 1.00 kV. Then, a 0.500- T magnetic field causes the ions to follow a circular path. Determine the radius of the path.

SUMMARY

In this session, you have learnt that:

1. A moving charge or current creates a magnetic field in the surrounding space in addition to its electric field. The field by which magnetic interaction occurs is also called magnetic field denoted by the symbol $\vec{B}$

2. The Lorentz force on a moving charge q is $\vec{F} = q\vec{E} + q\vec{V}\times\vec{B}$. $\vec{F}$ has the direction of $\vec{V}\times\vec{B}$ if q is positive while $\vec{F}$ has the direction of $-\vec{V}\times\vec{B}$ if q is negative

3. A charged particle in a uniform magnetic field traces a circular path of radius

$$r = \frac{mV}{qB} = \frac{P}{qB}$$

4. The period of motion of a charged particle in a constant magnetic field is given by:

$$T = \frac{2\pi r}{V} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$$

5. The frequency $f = \frac{1}{T} = \frac{qB}{2\pi m}$ is called the **cyclotron frequency**
6. Crossed electric and magnetic fields can be used as a velocity selector. The electric and magnetic forces exactly cancel when $V = \frac{E}{B}$

Study Session 10: Magnetic Fields and Magnetic Forces of / on Current Carrying Conductors

Expected Duration: 1 week or 2 contact hours

Introduction

Moving charges experience a force in a magnetic field. If these moving charges are in a wire, that is, if the wire is carrying a current, the wire will experience a force. In the last session, we studied the motion of charged particles in a uniform magnetic field. In this chapter, you will learn that a current-carrying wire generates a magnetic field and the magnetic field exerts a force on the current –carrying wire. However, before we discuss the force exerted on a current by a magnetic field, we first examined the magnetic field generated by an electric current.

Learning Outcomes

When you have studied this session, you should be able to explain:

10.1 Generation of a Magnetic Field

10.2 The Biot- Savart Law

10.3 The Magnetic Field at the Centre of a Circular-Current Loop

10.4 The Magnetic Field Around a Long Straight Wire

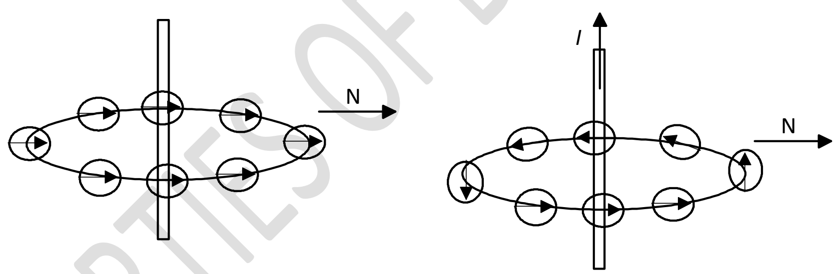
10.5 Ampere's Circuital Law

10.6 Force on a Current-Carrying Conductor in an External Magnetic Field

10.1 Generation of a Magnetic Field

Although magnets had been known for hundreds of years, it was not until 1820, that Hans Christians Oersted (1771-1851) discovered a relation between electric current and magnetic fields. If a series of compasses are placed around a wire that is not carrying a current, all the compasses will point toward the north, the direction of the earth's magnetic field, as shown in Figure 10.1. However, if, a current I passes through the wire, the compass needles will no longer point to the north. Instead, they point in a direction which is everywhere tangential to a circle drawn around the wire, passing through each compass, as shown in figure 10.1(b). Because a compass always align itself in the direction of a magnetic field, the current in the wire has created a circular magnetic field directed anticlockwise around the wire. If the direction of the current is reversed, the direction of the magnetic field will also be reversed and the compasses will point in a clockwise direction. The direction of the magnetic field around a long straight wire is determined by the "Right Hand Rule". In "*Right Hand Rule", your thumb points in the direction of the current while your fingers will curl around the wire, pointing in the direction of the magnetic field produced.*

This observation that electric currents can create a magnetic field was responsible for linking the then two independent sciences of electricity and magnetism into the one unified science of electromagnetism.



(a) No current in wire (b) Current in wire

Figure 10.1 The creation of a magnetic field by an electric current

10.2 The Biot – Savart Law

The definition of the magnetic forced showed that two moving charges experience a magnetic force. In other words, a moving charge produces a magnetic field which results in a magnetic force acting on all charges moving in this field.

A current flowing through a wire is equivalent to a collection of electrons moving with a certain velocity along the direction of the wire. Each of the moving electrons produces a magnetic field. The Biot-Savart law relates the amount of magnetic field dB at the position r produced by a small element dl, of a wire carrying a current I and is given by:

$$dB = \frac{\mu I}{4\pi} \frac{dl \times r}{r^3} \tag{10.1}$$

and is shown in Figure 10.2. μ is a constant called the permeability of the medium.

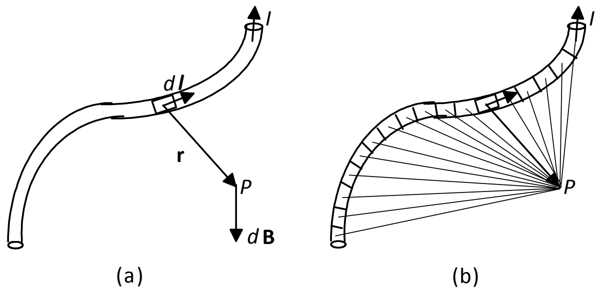


Figure 10.2 The magnetic field produced by a current element.

In a vacuum or air, it is called the permeability of free space, and is denoted by μ_0, where

$$\mu_0 = 4\pi \times 10^{-7}\ T\,m\,A^{-1}$$

Equation (10.1) is called the Biot-Savart law.

10.3 The Magnetic Field at the Centre of a Circular Current Loop

Biot-Savart law can be used to determine the magnetic field at the centre of a circular current loop shown in Figure 10.3. A small element of the wire *dl* produces an element of magnetic field *dB* at the centre of the wire given by equation (10.2) as:

$$dB = \frac{\mu_0 I}{4\pi} \frac{d\,l \times r}{r^3} \tag{10.2}$$

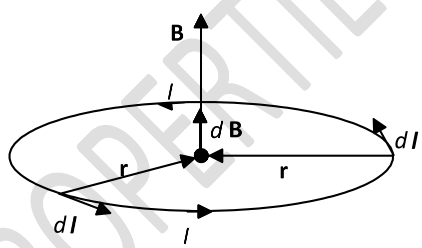


Figure 10.3 The magnetic field at the centre of a circular current loop

From the nature of the vector cross product, $d\,l \times r$, and hence *dB* points upward at the centre of the circle for every current element as shown in figure 10.3. The total magnetic field **B** which is the sum of all the *dB'*s, must also point upward at the centre of the loop. Therefore, the magnetic field at the centre of the current loop is perpendicular to the plane formed by the loop and points upward. Since the direction of the vector is now known, the total magnetic field gives:

$$B = \int dB \tag{10.3}$$

The magnitude of dB in equation (10.2) gives:

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl\, r \sin\theta}{r^3} \tag{10.4}$$

Since dl is perpendicular to r, the angle θ is equal to 90^0, therefore, dB becomes:

$$dB = \frac{\mu_0 I\, dl}{4\pi r^2} \tag{10.5}$$

Replacing equation (10.5) into equation (10.3) gives the magnitude of the magnetic field as:

$$B = \int dB = \int \frac{\mu_0 I\, dl}{4\pi r^2} \tag{10.6}$$

Since the loop is a circle of constant radius r and $\frac{\mu_0 I}{4\pi}$, is a constant, these terms can be factorized out of the integral to yield:

$$B = \frac{\mu_0 I}{4\pi r^2} \int dl \tag{10.7}$$

But the summation of all the *dl's* is simply the circumference of the wire.

$$\int dl = 2\pi r \tag{10.8}$$

Therefore, equation (10.7) becomes:

$$B = \frac{\mu_0 I}{4\pi r^2}(2\pi r) = \frac{\mu_0 I}{2r} \tag{10.9}$$

Equation (10.9) gives the magnetic field at the centre of a circular current loop. The magnetic field at the centre of the circular current loop is directly proportional to the current I –the larger the current, the larger the magnetic field; and inversely proportional to the radius r of the loop – the larger the radius, the smaller the magnetic field. If there are N turns of wire constituting the loop, the magnetic field at the centre is:

$$B = \frac{\mu_0 N I}{2r} \tag{10.10}$$

The magnetic field found in this way is the magnetic field at the centre of the current loop. The magnetic field all around the loop is shown in figure 10.4. It is observed that it resembles the magnetic field of a bar magnet, where the top of the loop would be the north pole.

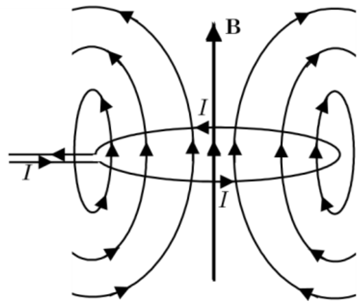


Figure 10.4 The magnetic field of a current loop

10.4 The Magnetic Field Around a Long Straight Wire

The Biot-Savart law can be used to determine the magnetic field at a distance R away from a long straight wire carrying a current *I* as shown in figure 10.5. The wire lies along the y-axis and is carrying a current in the positive y-direction as shown. A small element *dl* of the current carrying wire causes a small element of magnetic field *dB* at the point P given by the Biot-Savart law as:

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl \times r}{r^3} \qquad (10.11)$$

The total magnetic field at the point P will be the sum or integrals of all these *dB's* and is given by equation (10.12) as:

$$B = \int dB \qquad (10.12)$$

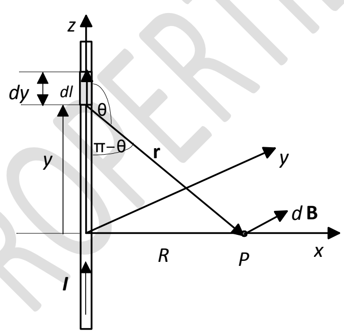


Figure 10.5 The magnetic field around a long straight wire by applying Biot-Savart law.

Vector *dl* is rotated into *r* in order to determine the direction of the cross product term $dl \times r$ and *dB.* It is noticed that it points into the plane of the paper at the point P. Since *dB* is always into the page, **B** will also be into the page, and we only have to deal with the magnitude of *dB* in the integration. The magnetic field **B** can be determined from:

$$B = \int dB = \int_{-\infty}^{\infty} \frac{\mu_0 I}{4\pi} \frac{dl \sin\theta}{r^2} \qquad (10.13)$$

The long straight wire is assumed to go from $y = -\infty$ to $y = +\infty$ and these are now the limits of integration. The element *dl* will now be represented by *dy* since *y* is the variable we are integrating over. Equation (10.13) becomes:

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta \, dy}{r^2} \qquad (10.14)$$

The variables θ, r, and y are not independent, but are related from the geometry of figure 10.5 as:

$$r = \sqrt{y^2 + R^2} \qquad (10.15)$$

and from trigonometric identity

$$\sin\theta = \sin(\pi - \theta)$$

Hence, $\sin\theta = \sin(\pi - \theta) = \dfrac{R}{\sqrt{y^2 + R^2}}$ (10.16)

Substituting equation (10.15) and equation (10.16) into equation (10.14) yields:

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta \, dy}{r^2} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{R}{\sqrt{y^2 + R^2}} \bullet \frac{dy}{\left(y^2 + R^2\right)}$$

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{R \, dy}{\left(y^2 + R^2\right)^{\frac{3}{2}}} = \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{\infty} \left(y^2 + R^2\right)^{\frac{-3}{2}} dy \qquad (10.17)$$

Because of the symmetry of the problem, we can integrate from $y = 0$ to $y = +\infty$, instead of integrating from $y = -\infty$ to $y = +\infty$, by doubling the value of the integral. That is,

$$\int_{-\infty}^{\infty} dy = 2\int_{0}^{\infty} dy$$

Equation (10.17) can be written as:

$$B = \frac{2\mu_0 I R}{4\pi} \int_{0}^{\infty} \left(y^2 + R^2\right)^{\frac{-3}{2}} dy \qquad (10.18)$$

But from table of integrals, we find

$$\int_0^\infty \left(y^2 + R^2\right)^{\frac{-3}{2}} dy = \frac{y}{R^2\sqrt{y^2 + R^2}} \tag{10.19}$$

Substituting equation (10.19) into equation (10.18) yields:

$$B = \frac{\mu_0 I R}{2\pi}\left[\frac{y}{R^2\sqrt{y^2 + R^2}}\right]_0^\infty \tag{10.20}$$

If we were to place the limits of integration into the present form of equation (10.20) , we would obtain an indeterminate form. Hence, we now divide both numerator and denominator by *y* to give:

$$B = \frac{\mu_0 I R}{2\pi R^2}\left[\frac{1}{\sqrt{\frac{y^2}{y^2} + \frac{R^2}{y^2}}}\right]_0^\infty = \frac{\mu_0 I}{2\pi R}\left[\frac{1}{\sqrt{1 + \frac{R^2}{y^2}}}\right]_0^\infty = \frac{\mu_0 I}{2\pi R}\left[\frac{1}{\sqrt{1+\frac{R^2}{\infty}}} - \frac{1}{\sqrt{1+\frac{R^2}{0}}}\right]$$

$$B = \frac{\mu_0 I}{2\pi R}\left[\frac{1}{\sqrt{1+0}} - \frac{1}{\sqrt{1+\infty}}\right] = \frac{\mu_0 I}{2\pi R}[1-0]$$

$$B = \frac{\mu_0 I}{2\pi R} \tag{10.21}$$

Equation (10.21) gives the value of the magnetic field **B** at a distance R from a long straight wire carrying a current *I*.

10.5 Ampere's Circuital Law

In session 10.4, you learnt that the Biot-Savart law can be used to determine the magnetic field for different current distributions in which many of the derivations require vector integrations. Another simpler techniques for determining the magnetic fields, when symmetry is appropriate is Ampere's Circuital Law. *Ampere's Law states that along any arbitrary path encircling a total current I_{total} ,the integral of the scalar product of the magnetic field **B** with the element of length dl of the path, is equal to the product of the permeability μ_0 and the total current I_{total} enclosed by the path.* That is:

$$\oint B.dl = \mu_0 I_{total} \qquad (10.22)$$

Ampere's law is a fundamental law that is based on experiments and cannot be derived.

10.6 Force on a Current-Carrying Conductor in an External Magnetic Field

If a wire carrying a current *I* is placed in an external magnetic field **B** as shown in figure 10.6, a force will be found to act on the wire. This force is the magnetic force acting on a charged particle in a magnetic field.

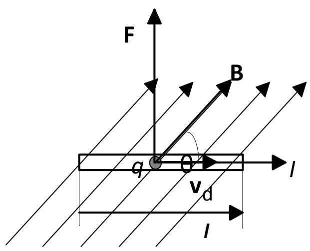


Figure 10.6 Force on a current-carrying wire in an external magnetic field.

If the wire is carrying a current, then there are charges in motion within the wire. These charges will be moving with a drift velocity V_d in the direction of the current flow. Any one of these charges *q* will experience the force

$$F_q = qV_d \times B \qquad (10.23)$$

This force on an individual charge will cause the charge to interact with the lattice structure of the wire, exerting a force on the lattice and hence on the wire itself. The drift velocity of the moving charge can be written as:

$$V_d = \frac{l}{t} \qquad (10.24)$$

where *l* is a small length of the wire in the direction of the current flow and is shown in figure 10.6, and *t* is the time. Replacing this drift velocity in equation (10.23) gives:

$$F_q = q\left(\frac{l}{t}\right) \times B = \left(\frac{q}{t}\right) l \times B \qquad (10.25)$$

The net force on the wire is the sum of the individual forces associated with each charge carrier, that is,

$$F = \sum_q \sum F_q = \sum_q \left(\frac{q}{t}\right) l \times B$$

But $\sum_q \left(\frac{q}{t}\right)$ is equal to all the charges passing through a plane of the wire per unit time and is defined to be the current in the circuit, *I*. Hence *a wire carrying a current I in any external magnetic field* ***B,*** *will experience a force given by:*

$$F = I\,l \times B \qquad (10.26)$$

The force is again given by a cross product term, and the direction of the force is found from $l \times B$. If *l* is in the direction of the current and **B** is pointing into the page in figure 10.6, $l \times B$ is a vector that points upward. If the direction of the current flow is reversed, *l* would be reversed and $l \times B$ would then point downward. *The magnitude of the force is determined from equation (10.24) as:*

$$F = I\,l\,B\sin\theta \qquad (10.27)$$

where θ is the angle between l and B. Solving equation (10.27) is gives another set of units for the magnetic field, namely:

$$B = \frac{F}{I\,l}$$

Thus, 1 Tesla $= \dfrac{Newton}{Ampere \times metre} \quad \Rightarrow 1\,T = 1\,N\,Am^{-1}$

Worked Examples

1. A 20.0 cm wire carrying a current of 10.0 A is placed in a uniform magnetic field of 0.300 T. If the wire makes an angle of 40.0^0 with the vector **B**. Determine the direction and magnitude of the force on the wire.
   Solution:
   The direction of the force is obtained from the equation below:

   $$F = I\,l \times B$$

   In rotating the vector *l* towards the vector **B** in the cross product, the thumb points upward, indicating that the direction of the force is also upward. The magnitude of the force is:

   $$F = I\,l\,B\sin\theta$$

   $$F = (10.0\,A)(0.200\,m)(0.300\,T)\sin 40^0$$

   $$F = 0.386\,N$$

2. Determine the magnetic field at the centre of a circular current loop of 0.500 m radius, carrying a current of 7.00 A.
   Solution:
   The magnetic field at the centre of the loop is given by:

$$B = \frac{\mu_0 \ I}{2r} = \frac{\left(4\pi \times 10^{-7} \ T\,m\,A^{-1}\right)\left(7.00\,A\right)}{2\left(0.500\,m\right)}$$

$$B = 8.80 \times 10^{-6}\,T$$

3. Determine the magnetic field at the centre of a circular current loop of 10 turns, with a radius of 5.00 cm carrying a current of 10.0 A.
   Solution:
   The magnetic field is given by:

$$B = \frac{\mu_0 \,N\,I}{2r} = \frac{\left(4\pi \times 10^{-7} \ T\,m\,A^{-1}\right)\left(10\right)\left(10.0\,A\right)}{2\left(0.050\,m\right)}$$

$$B = 1.26 \times 10^{-3}\,T$$

4. A long straight wire is carrying a current of 15.0 A. Calculate the magnetic field 30.0 cm from the wire.
   Solution:
   The magnetic field around the wire is given by:

$$B = \frac{\mu_0 \,I}{2\pi R} = \frac{\left(4\pi \times 10^{-7} \ T\,m\,A^{-1}\right)\left(15.0\,A\right)}{2\pi\left(0.30\,m\right)}$$

$$B = 1.00 \times 10^{-5}\,T$$

5. A toroid has an inner radius of 10.0 cm and an outer radius of 20.0 cm and carries 500 turns of wire. If the current in the toroid is 5.00 A. Determine the minimum and maximum values of the magnetic field inside the toroid.
   Solution:
   The minimum value of the magnetic field B within the toroid occurs for the maximum value of r , given by:

$$B = \frac{\mu_0 \ N\,I}{2\pi R} = \frac{\left(4\pi \times 10^{-7} \ T\,m\,A^{-1}\right)\left(500\right)\left(5.00\,A\right)}{2\pi\left(0.200\,m\right)}$$

$$B = 0.0025T$$

   The maximum value of the magnetic field B within the toroid occurs for the minimum value of r, and is given by:

$$B = \frac{\mu_0 \ N\,I}{2\pi R} = \frac{\left(4\pi \times 10^{-7} \ T\,m\,A^{-1}\right)\left(500\right)\left(5.00\,A\right)}{2\pi\left(0.100\,m\right)}$$

$$B = 0.0050\,T$$

Self – Assessment Questions (SAQs)

1. An electric power line carries a current of 1400 A in a location where the earth's magnetic field is 5.0 x 10^{-5} T. The line makes an angle of 75^0 with respect to the field. Determine the magnitude of the magnetic force on a 120-m length of line.

2. In the diagram below, a 6.00 m long wire carrying a current of 120 A is immersed in a uniform magnetic field of magnitude 0.200 T and width 3.50 m. Determine the magnetic force on the wire.

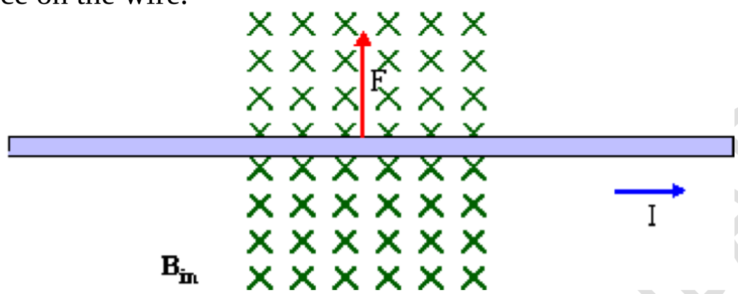


Fig.10.7

3. A long straight wire of linear mass density 20 g/m is immersed in a constant magnetic field B = 3 T, as shown in Fig 10.8 Determine the current *I* that would be required by the wire to be suspended.

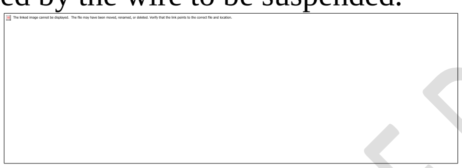


Fig. 10.8: Long straight wire in a magnetic field.

4. In the two long straight wires shown in Figure 10.9, each wire carries a current of 5A in the opposite directions and are separated by a distance *d* = 30 cm. Find the magnetic field a distance *l* = 20 cm to the right of the wire on the right hand side.

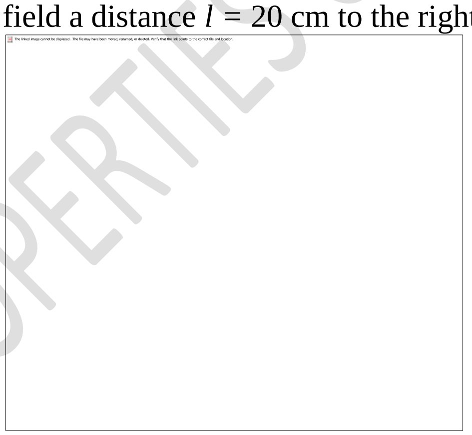


Fig. 10.9: Magnetic field of two long straight wires

5. In the arrangement shown in Fig.10.10, the long straight wire carries a current of $I_1 = 5A$.This wire is at a distance *d* = 0.1 m away from a rectangular loop of dimensions *a* = 0.3 m and *b* = 0.4 m which carries a current $I_2 = 10A$. Determine the net force exerted on the rectangular loop by the long straight wire.

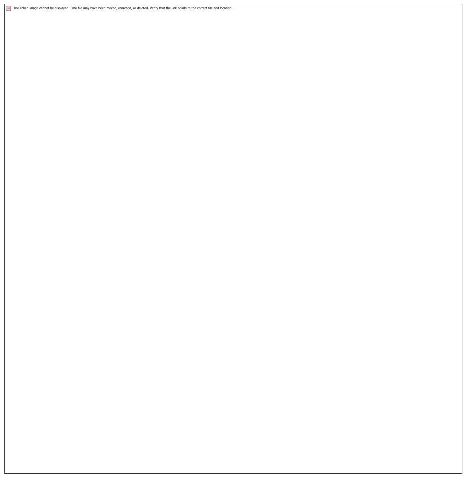


Fig. 10.10: Force on a current loop due to a long straight wire.

SUMMARY

In this session, you have learnt that:

1. Magnetic field is the field of force experienced by a magnetized body or a current carrying wire. The SI unit of magnetic field is Tesla (T).
2. The direction of the magnetic field around a long straight wire is determined by the "Right Hand Rule". In "Right Hand Rule", your thumb points in the direction of the current while your fingers will curl around the wire, pointing in the direction of the magnetic field produced.
3. The Biot-Savart law relates the amount of magnetic field *dB* at the position *r* produced by a small element *dl*, of a wire carrying a current *I* and is given by:
$$dB = \frac{\mu I}{4\pi} \frac{dl \times r}{r^3}$$
4. The magnetic field at the centre of a circular current loop is given by:
$$B = \frac{\mu_0 I}{2r}$$
5. The magnetic field around a long straight wire is given by:
$$B = \frac{\mu_0 I}{2\pi R}$$
6. Ampere's Circuital law states that along any arbitrary path encircling a total current I_{total}, the integral of the scalar product of the magnetic field **B** with the element of length *dl* of the path, is equal to the product of the permeability μ_0 and the total current I_{total} enclosed by the path. That is:

$$\oint B.dl = \mu_0 I_{total}$$

7. Force on a charged particle in an external magnetic field is given by:

$$F_q = qV_d \times B = qV_d B\sin\theta$$

8. Force on a current-carrying conductor in an external magnetic field is given by:

$$F = Il \times B = IlB\sin\theta$$

Study Session 11: Concept of Electromagnetic Induction

Expected Duration: 1 week or 2 contact hours

Introduction

In the last session, you learnt that a current in a wire produces a magnetic field. Michael Faraday (1791-1867) also experimented with electric and magnetic phenomena and discovered that a changing magnetic field produced an induced emf (voltage –sources of electrical energy). The process whereby a magnetic field can produce a current is called electromagnetic induction. Faraday's law of electromagnetic induction of electromagnetic induction is one of the important laws of Physics. This phenomenon is the scientific principal that is the basis for many practical devices such as transformer, communication and data storage device (reading computer memory), electronic devices, alternators and generators. Generators produce large quantities of electrical energy required for our modern society to function. In this session, you will learn the method of producing electricity by varying magnetic field.

Learning Outcomes

When you have studied this session, you should be able to explain the:

11.1 Magnetic Flux

11.2 Motional Emf and Faraday's Law of Electromagnetic Induction

11.3 Lenz's Law of Electromagnetic Induction

11.4 Mutual Inductance and Self Inductance

11.5 Energy Stored in the Magnetic Field of an Inductor

11.1 Magnetic Flux

In session 3, electric flux was defined as a quantitative measure of the number of electric field lines passing normally through a surface. Similarly, the magnetic flux ϕ can be defined as a quantitative measure of the number of magnetic field lines **B** crossing a particular surface area **A** normally. Figure 11.1(a) shows a magnetic field **B** passing through a portion of a surface area **A**. The magnetic flux is defined to be

$$\phi = B \bullet A \tag{11.1}$$

and is a quantitative measure of the number of lines **B** that pass normally through the surface area **A**. The number of lines represent the strength of the field. The vector **B**, at the point P of figure 11.1(a), can be resolved into the components, $B_\perp$ the perpendicular component to the surface, and B_X the parallel components. The perpendicular component is given by

$$B_\perp = B\cos\theta$$

While the parallel component is given by

$$B_X = B\sin\theta$$

The parallel component B_X lies in the surface itself and therefore does not pass through the surface, while the perpendicular component $B_\perp$ completely passes through the surface at the point P. The product of the perpendicular component $B_\perp$ and the area A

$$B_\perp \mathrm{A} = (B\cos\theta)A = BA\cos\theta = B \bullet A = \phi \tag{11.2}$$

is therefore a quantitative measure of the number of lines of **B** passing normally through the entire surface area **A.** θ is the angle between the magnetic field and area vector. If θ in equation (11.2) is zero, then **B** is parallel to the vector **A** and all the lines of **B** pass normally through the surface area A, as shown in figure 11.1(b). If the angle θ in equation (11.2) is 90^0 then **B** is perpendicular to the area vector **A,** and none of the lines pass through the surface A as shown in figure 11.1(c). The SI unit of magnetic flux is Weber (Wb). It can be deduced from equation (11.2) that 1 Tesla = 1 Weber/ m^2

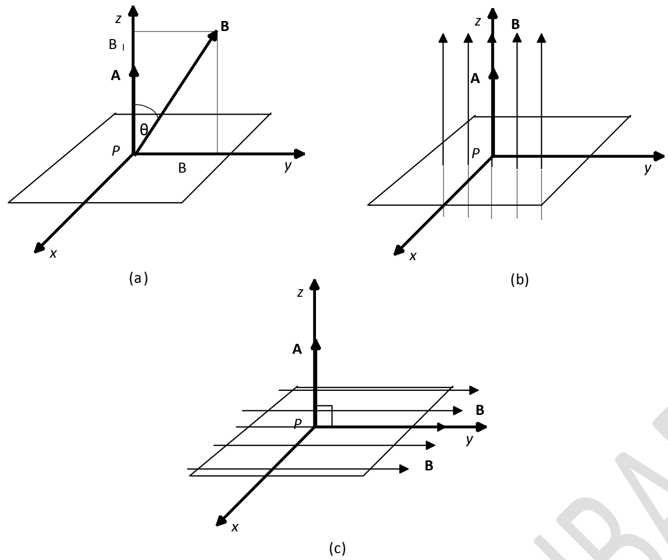
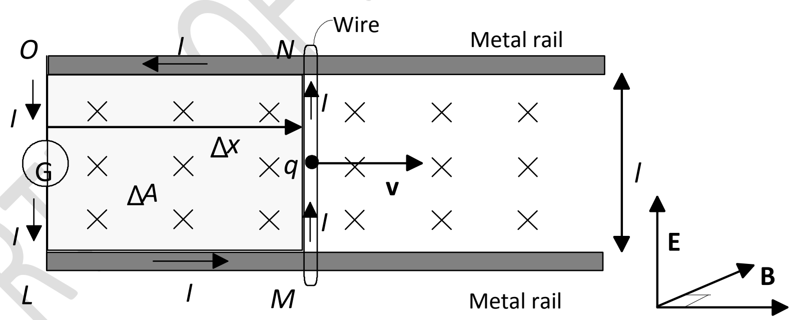


Figure 11.1 The Magnetic Flux

11.2 Motional Emf and Faraday's Law of Electromagnetic Induction

Figure 11.2 shows two parallel metal rails separated by a distance l . A metal wire rests on the two rails. A uniform magnetic field **B** is applied such that its direction is into the paper as shown. A galvanometer G is connected across the two rails. The galvanometer



(a) (b) **v**

Figure 11.2 Motional emf

reads zero indicating that there is no current in the circuit which consists of the rails and wires, that is, the circuit is the electrical path designated LMNOL in figure 11.2(a). The metal wire MN is now pulled along the rails at a velocity v to the right. The galvanometer now indicates that a current is flowing in the circuit. This implies that the motion of the wire through the magnetic field generated an electric current. The electric current generated can be explained by assuming that as the wire MN is moved to the right, any charge q within the wire experiences the force

$$F = qv \times B \qquad (10.21)$$

as shown in section 10.6. If both sides of equation (10.21) are divided by q, gives:

$$\frac{F}{q} = v \times B \qquad (11.3)$$

But an electrostatic field was originally defined as:

$$E = \frac{F}{q_0}$$ (2.1.1) where F was the force acting on a test charge q_0 placed at rest in the electrostatic field. The induced electric field **E** can be defined by equation (11.3) as:

$$E = \frac{F}{q} = v \times B \tag{11.4}$$

This is quite different from the electrostatic field. The induced electric field exists only when the charge is in motion at a velocity v . When v = 0, the induced electric field will also be zero (see equation 11.4). The induced electric field is the cause of the electric current in the wire. The cross product $v \times B$ shows the direction of the induced electric field in figure 11.2(b) and hence the direction that a positive charge q within the wire, will move. Therefore, the direction of the current will be in the direction $M \Rightarrow N \Rightarrow O \Rightarrow L \Rightarrow M$ as shown in figure 11.2(b). The magnitude of the induced electric field van be obtained from equation (11.4) and the definition of the cross product as:

$$E = vB\sin\theta \tag{11.5}$$

The angle between v and B is 90^0 and the induced electric field is therefore:

$$E = vB \tag{11.6}$$

It was shown in session 4 that for a uniform electric field that:

$$E = \frac{V}{d}$$

where V is the potential difference between two points and d and d is the distance between them. For the connecting wire MN, the induced electric field within the wire can be assumed to be uniform and the induced potential difference V between M and N is called an induced emf designated by ε. The distance d between M and N is the length l of the wire. Therefore, the induced electric field can be written as:

$$E = \frac{\varepsilon}{l} \tag{11.7}$$

Equating equation (11.6) to equation (11.7) gives:

$$\frac{\varepsilon}{l} = vB$$

The induced emf ε in the wire is therefore:

$$\varepsilon = vBl \tag{11.8}$$

If the circuit has a resistance R , then there is an induced current in the circuit given by Ohm's law as:

$$I = \frac{\varepsilon}{R} \tag{11.9}$$

This is the current that is recorded by the galvanometer.

The induced emf can also be derived by noting that the speed v of the wire is $v = \frac{dx}{dt}$

where dx is the distance the wire moves to the right in the time dt. The product of v and l in equation (11.8) can then be written as:

$$vl = \left(\frac{dx}{dt}\right)l \qquad (11.10)$$

But $$(dx)l = dA \qquad (11.11)$$

is the area of the loop swept out as the wire is moved to the right and is shown in figure 11.2(a). Substituting equation (11.11) into equation (11.10) gives:

$$vl = \frac{dA}{dt} \qquad (11.12)$$

and the induced emf in equation (11.8) becomes:

$$\varepsilon = B\frac{dA}{dt} \qquad (11.13)$$

Recall that the magnetic flux $\phi = B \bullet A$ (11.2)

For a constant magnetic field **B**, the change in magnetic flux is:

$$d\phi = B \bullet dA \qquad (11.14)$$

The rate at which the magnetic flux changes with time is:

$$\frac{d\phi}{dt} = B \bullet \frac{dA}{dt} \qquad (11.15)$$

In figure 11.2(a), **B** is into the paper, while *d*A, the change in area vector, is out of the paper, hence the angle between **B** and *d*A is 180^0. Using this fact in equation (11.15) gives:

$$\frac{d\phi}{dt} = B\frac{dA}{dt}\cos 180^0 = -B\frac{dA}{dt} \qquad (11.16)$$

Therefore,

$$B\frac{dA}{dt} = -\frac{d\phi}{dt} \qquad (11.17)$$

Combining equation (11.13) and (11.17) gives an important relationship known as Faraday's law, that is,

$$\varepsilon = -\frac{d\phi}{dt} \qquad (11.18)$$

Therefore, *Faraday's law of electromagnetic induction states that whenever the magnetic flux changes with time, there will be an induced emf.*

In a case of a closely wound coil of N turns, change of magnetic flux associated with each turn is the same. The expression of the total induced emf is given by:

$$\varepsilon = -N\frac{d\phi}{dt} \qquad (11.19)$$

11.3 Lenz’s Law of Electromagnetic Induction

Faraday’s law of electromagnetic induction was derived in section 11.2 as:

$$\varepsilon = -\frac{d\,\phi}{d\,t} \tag{11.18}$$

The effect of minus sign in Faraday’s law gives rise to a relation known as Lenz’s law .Lenz’s law states that: *The direction of an induced emf is such that any current it produces, always opposes, through the magnetic field of the induced current, the change inducing the emf.*

Or

The polarity of induced emf is such that it tends to produce a current which opposes the change in magnetic flux that produced it.

Figure 11.3 Illustration of Lenz’s law

In the above figure 11.3(a), when the North-pole of a bar magnet is being pushed towards a closed coil, the magnetic flux through the coil increases. Hence current is induced in the coil in such a direction that it opposes the increase in flux. This is possible only if the current in the coil is in a counter-clockwise direction with respect to an observer situated on the side of the magnet.

Similarly, in the above figure 11.3(b), if the North-pole of the bar magnet is being withdrawn from the coil, the magnetic flux through the coil will decrease. To counter this decrease in magnetic flux, the induced current in the coil flows in clockwise direction and its South-pole faces the receding North-pole of the bar magnet. This would result in an attractive force which opposes the motion of the magnet and the corresponding decrease in flux. The above illustration shows that Lenz's law complies with the principle of conservation of energy. In this case, when the N-pole of a bar magnet is pushed into a coil as shown in figure 11.3, the direction of induced current in the coil will act as N-pole. So, work has to be done against the magnetic repulsive force to push the magnet into the coil. The electrical energy is produced in the coil at the expense of the work done.

11.4 Mutual Inductance and Self Inductance

An electric current can be induced in a coil by magnetic flux change produced by another coil in its vicinity of magnetic flux change produced by the same coil. However, in both cases, the magnetic flux through the coil is proportional to the current. That is, $\phi \propto I$ For a closely wound coil of N turns, the same magnetic flux is linked with all the turns. When the magnetic flux ϕ through the coil changes, each turn contributes to the induced emf. Therefore, a term called *flux linkage* is used which is equal to $N\phi$ for a closely wound coil and in such a case $N\phi \propto I$. The constant of proportionality is called *inductance*. Inductance depends on the geometry of the coil and intrinsic material properties. Inductance is a scalar quantity and its SI unit is henry (H). It has the dimensions of $\left[M\,L^2\,T^{-2}\,A^{-2}\right]$.

11.4.1 Mutual Inductance

Consider the two coaxial solenoids of the same size shown in figure 11.4. If the current I_1 in coil 1, is changed with time, this will change the magnetic field B_1 of solenoid 1 with time. But solenoid 2 is in the magnetic field of coil 1 and any change in B_1 will cause an induced emf in coil 2. From Faraday's law of induction, this can be stated as:

$$\varepsilon_2 = -N_2\frac{d\phi}{dt} = -N_2\frac{d}{dt}\left(B_1\,A\right) = -N_2\,A\frac{d\,B_1}{dt} \qquad (11.20)$$

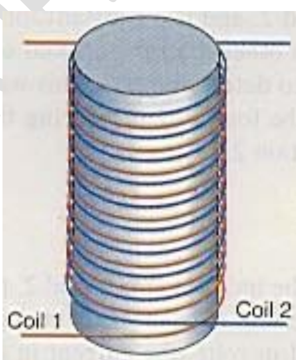


Figure 11.4 : Two coaxial solenoids

The magnetic field inside solenoid 1 is given by:

$$B_1 = \mu_0 n_1 i_1 = \mu_0 \frac{N_1}{l} i_1 \qquad (11.21)$$

when i_1 changes with time, B_1 will change as:

$$d B_1 = \mu_0 \frac{N_1}{l} di_1 \qquad (11.22)$$

Substituting equation (11.22) into equation (11.20) gives:

$$\varepsilon_2 = -N_2 A\left(\mu_0 \frac{N_1}{l}\frac{d i_1}{d t}\right) = -\mu_0 A \frac{N_1 N_2}{l}\frac{d i_1}{d t} \qquad (11.23)$$

Let us *define the quantity*

$$\frac{\mu_0 N_1 N_2}{l} = M \qquad (11.24)$$

to be the coefficient of mutual induction for the coaxial solenoids. Faraday's law now becomes:

$$\varepsilon_2 = -M \frac{d i_1}{d t} \qquad (11.25)$$

Equation (11.25) says that changing the current i_1 in coil 1 with time induces an emf in coil 2. From equation (11.25), M is only a function of the geometry of solenoids 1 and 2, and is a constant for particular set of solenoids chosen. The coefficient of mutual inductance M is defined as:

$$M = -\frac{\varepsilon_2}{\frac{d i_1}{dt}} \qquad (11.26)$$

Similarly, the induced emf in coil 1 caused by a changing current in coil 2 is given by:

$$\varepsilon_1 = -M \frac{d i_2}{d t} \qquad (11.27)$$

11.4.2 Self Inductance

In the previous sub-section, we considered the magnetic flux in one solenoid due to current in the other. It is also possible that emf is induced in a single isolated coil due to change of magnetic flux through the coil by means of varying the current through the same coil. This phenomenon is called *self –induction* and can be explained by Faraday's law of electromagnetic induction.

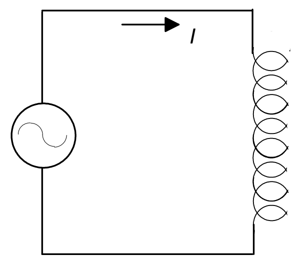


Figure 11.4 Self Induction

A single solenoid is shown in figure 11.4. If the two ends of the solenoid coil are attached to the AC Generator, the varying current in the coil will cause a varying magnetic field to exist in the coil. The magnetic field of a solenoid is given by:

$$B = \mu_0 n i \tag{11.28}$$

By changing the current i causes the magnetic field to change by

$$dB = \mu_0 n\, di \tag{11.29}$$

The induced emf is given by Faraday's law as:

$$\varepsilon = -N\frac{d\phi}{dt} = -N\frac{d(BA)}{dt}$$

Because the area A of the coil does not change, this becomes:

$$\varepsilon = -NA\frac{dB}{dt} \tag{11.30}$$

The changing magnetic field within the solenoid is given by equation (11.29) and substituting it into equation (11.30) gives:

$$\varepsilon = -NA\mu_0 n\frac{di}{dt} \tag{11.31}$$

The total number of turns N is related to n, the number of turns per unit length by

$N = nl$ where l is the length of the solenoid. Substituting this into equation(11.31) gives:

$$\varepsilon = -\left(\mu_0 \, A l \, n^2\right)\frac{d\,i}{d\,t} \tag{11.32}$$

The coefficient of $\frac{di}{dt}$ is a constant which depends only upon the geometry of the solenoid coil. This constant is called the *self-inductance of the solenoid coil* and is designated by L

$$L = \mu_0 \, A l \, n^2 \tag{11.33}$$

The self- induced emf is now given by:

$$\varepsilon = -L\frac{di}{dt} \tag{11.34}$$

Equation(11.34) shows that changing the current in a coil will induce an emf in the coil, and the minus sign means that the induced emf will act to oppose the cause of the induced emf. Equation (11.34) can be used to define the self- inductance as:

$$L = -\frac{\varepsilon}{\frac{di}{dt}} \tag{11.35}$$

ε and $\frac{di}{dt}$ can be measured experimentally for any coil configuration and L can be determined from equation (11.35). L, the self-inductance is usually called the inductance of the coil measured in henry. A circuit element in which a self-induced emf accompanies a changing current is called an inductor.

11.5 Energy Stored in the Magnetic Field of an Inductor

It was shown in section 5.4 that energy can be stored in the electric field between the plates of a capacitor. In a similar manner energy is stored in the magnetic field of the coils of an inductor. When the switch in figure 11.5(a) is closed, the total applied voltage V is impressed across the ends of the solenoid coil. The initial current *i* is zero. In order for the current to increase an amount of charge *dq* must be taken out of the positive side of the battery and moved against the induced emf ε in the coil. The amount of work done by the battery in moving this small amount of charge is:

$$d\,W = (d\,q)\varepsilon \tag{11.36}$$

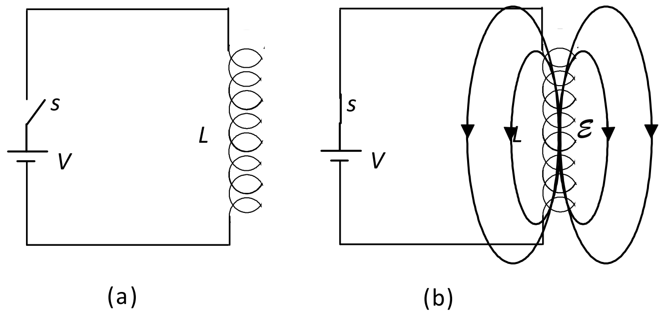


Figure 11.5: Energy stored in the magnetic field of an inductor

But the magnitude of the induced emf is given by equation (11.34) as:

$$\varepsilon = L\frac{di}{dt} \tag{11.37}$$

Therefore, the small amount of work done is:

$$dW = dq\,L\frac{di}{dt} = \frac{dq}{dt}L\,di = i\,L\,di \tag{11.38}$$

We have used the fact that $\frac{dq}{dt} = i$ (current). This current *i is* not constant, but varies with time. Therefore, the small amount of work done is not a constant, but varies with the current *I in* the circuit. The total work done is:

$$W = \int dW = \int_0^I L\,i\,di$$

$$W = \frac{1}{2}LI^2 \tag{11.39}$$

This work done W by the battery on the charge, shows up as potential energy of the charge. This energy is said to reside in the magnetic field of the coil has shown in figure 11.5(b) and is designated as U. Thus, the energy stored in the inductor is given by:

$$U = \frac{1}{2}LI^2 \tag{11.40}$$

This stored energy can also be expressed in terms of the magnetic field B by recalling that for a solenoid

$$L = \mu_0\,Al\,n^2 \tag{11.33}$$

and

$$B = \mu_0\,nI \tag{11.28}$$

Solving equation (11.28) for the current gives:

$$I = \frac{B}{\mu_0 n} \quad (11.41)$$

Substituting equation (11.33) and equation (11.41) into equation (11.40) gives:

$$U = \frac{1}{2}\frac{B^2 Al}{\mu_0} \quad (11.42)$$

Equation (11.42) gives the energy stored in the magnetic field of a solenoid.

The energy density is defined as the energy per unit volume and can be represented as:

$$u = \frac{U}{V} = \frac{1}{2}\frac{B^2 Al}{V\mu_0}$$

But the volume of the solenoid is

$$V = Al$$

Therefore,

$$u = \frac{1}{2}\frac{B^2}{\mu_0} \quad (11.43)$$

Equation (11.43) gives the magnetic energy density or the energy per unit volume that is stored in the magnetic field.

Worked Examples

1. The wire MN in figure 11.2 (a) moves with a velocity of $50\,cm\,s^{-1}$ to the right. If $l = 25.0\,cm$, $B = 0.250T$, and the total electric resistance of the circuit is 35.0Ω. Determine the:
   (a) Induced emf in the circuit
   (b) Current in the circuit

Solution:

(a) The induced emf in the circuit, $\varepsilon = vBl = (0.50\,ms^{-1})(0.250T)(0.25m)$

$$\varepsilon = 3.13 \times 10^{-2}\ V$$

(b) The current flowing in the circuit , $I = \frac{\varepsilon}{R} = \frac{3.13 \times 10^{-2}\ V}{35.0\ \Omega}$

$$I = 8.94 \times 10^{-4}\ A$$

2. A magnetic field of $5.00 \times 10^{-2}\,T$ passes through a plane 25.0 cm by 35.0 cm at an angle of 40^0 to the normal. Determine the magnetic flux passing through the plane.

Solution:
The area of the plane is:

$$A = (0.250\,m)(0.350\,m) = 8.75 \times 10^{-2}\ m^2$$

The magnetic flux, $\phi = BA\cos\theta = \left(5.00 \times 10^{-2}\ T\right)\left(8.75 \times 10^{-2}\ m^2\right)\cos 40^0$

$$\phi = 3.35 \times 10^{-3}\ Wb$$

3. The wire MN in figure 11.2(a) is fixed 10.0 cm away from the galvanometer wire OL. The magnetic field varies from 0 to 0.500 T in a time of $2.0 \times 10^{-3}\ s.$ If the resistance of the circuit is $35.0\,\Omega$. Calculate the:
(a)Induced emf
(b) current in the circuit while the magnetic field is changing with time
(c) induced emf if the magnetic field remains at a constant 0.500 T
Solution:
(a) From Faraday's law:

$$\varepsilon = -\frac{d\phi}{dt} = -B.\frac{dA}{dt} - A.\frac{dB}{dt}$$

Because the wire MN is fixed, the area of the loop does not change with time. That is, $dA = 0$.However, B is changing with time, and the induced emf is therefore,

$$\varepsilon = -A.\frac{dB}{dt}$$

The changing magnetic field is:

$$dB = B_f - B_i$$

Since the initial magnetic field $B_i = 0$, dB has the direction of B_f which is perpendicular to the paper and into the paper. The angle between A and dB is 180^0.

$$\varepsilon = -A\frac{dB}{dt}\cos 180^0 = A\frac{dB}{dt} = (0.100\,m)(0.250\,m)\left(\frac{0.500T - 0T}{2 \times 10^{-3}\ s}\right)$$

$$\varepsilon = 6.25V$$

(b) The induced current is:

$$I = \frac{\varepsilon}{R} = \frac{6.25V}{35.0\Omega} = 0.179\,A$$

(c) When the magnetic field remains constant at 0.500 T, there is no changing magnetic field, $dB = 0$, and there is no induced emf. That is:

$$\varepsilon = A\frac{dB}{dt} = 0$$

Thus, for a loop of wire of constant area, the only induced emf occurs when there is a changing magnetic field with time.

4. Determine the mutual inductance of two coaxial solenoids of 5.00 cm radius and 30.0 cm long, if one coil has 10 turns and the second has 1000 turns. If the current in the first coil changes by 2.00 A in 0.001 s. What is the induced emf in the second coil?
Solution:

The area of the solenoid, $A = \pi r^2 = \pi(0.0500\,m)^2$

$$A = 7.85 \times 10^{-3}\ m^2$$

(a) The mutual inductance ,M is given by:

$$M = \frac{\mu_0 N_1 N_2}{l} = \frac{(4\pi \times 10^{-7})(7.85 \times 10^{-3})(10)(1000)}{0.300}$$

$$M = 3.29 \times 10^{-4}\ H$$

(b) The induced emf in the second coil is:

$$\varepsilon_2 = -M\frac{d\,i_1}{d\,t} = -(3.29 \times 10^{-4})\frac{(2.00)}{0.001} = -0.658V$$

5. A battery is connected through a switch to a solenoid coil. The coil has 50 turns per centimetre, has a diameter of 10.0 cm and is 50.0 cm long. When the switch is closed, the current goes from 0 to its maximum value of 3.00 A in 0.002 s. Determine the inductance of the coil and the induced emf in the coil during this period.
Solution:
The cross-sectional area of the coil is:

$$A = \frac{\pi d^2}{4} = \pi\frac{(0.100\,m)^2}{4} = 7.85 \times 10^{-3}\ m^2$$

The inductance of the solenoid is:

$$L = \mu_0 A l n^2 = (4\pi \times 10^{-7})(7.85 \times 10^{-3})(0.500)(50)^2$$

$$L = 0.19H$$

The induced emf in the coil is:

$$\varepsilon = -L\frac{di}{dt} = -(0.19)\left(\frac{3.00 - 0}{0.002}\right)$$

$$\varepsilon = -185V$$

The minus sign on ε, indicates that it is opposing the battery voltage V

Self – Assessment Questions (SAQs)

1. A uniform magnetic field is normal to the plane of a circular loop 10cm in diameter and made of copper wire of diameter 2.5 mm. The resistivity of the copper wire is $1.68 \times 10^{-8}\ \Omega m$.
   (a) Calculate the resistance of the wire
   (b) At what rate must the magnetic field change with time if an induced current of 10A is to appear in the loop?
2. An electric generator consists of 100 turns of wire formed into a rectangular loop 50.0 cm by 30.0 cm, placed entirely in a magnetic field with magnitude B = 3.50 T. Determine the maximum value of emf when the loop is spun at 1000 rev/min about an axis perpendicular to B.
3. A UHF television loop antenna has a diameter of 11 cm. The magnetic field of a TV signal is normal to the plane of the loop and, at one instant of time, its magnitude is changing at the rate 0.16T/s. If the magnetic field is uniform, calculate the induced emf in the antenna.
4. A metal rod is forced to move with constant velocity along two parallel metal raids, connected with a strip of metal at one end. A magnetic field, B = 0.350 T points out of the page.
   (a) If the rails are separated by 25.0 cm and the speed of the rod is 55.0 cm/s, determine the generated emf .
   (b) If the rod has a resistance of 18.0 Ω and the rails and connector have negligible resistance, what is the current on the rod?
   (c) At what rate is energy being transferred to thermal energy?
5. A square loop of sides 10 cm and resistance 0.5 **Ω** is placed vertically in the east-west plane. A uniform magnetic field of 0.10 T is set across the plane in the north-east direction. The magnetic field is decreased to zero in 0.70s at a steady rate. Determine the magnitude of induced emf and current during this time interval.
6. A circular coil of radius 10 cm, 500 turns and resistance 2Ω is placed with its plane perpendicular to the horizontal component of the earth's magnetic field. It is rotated about its vertical diameter 180^0 in 0.25 s. Determine the magnitude of the emf and current induced in the coil. Horizontal component of the earth's magnetic field at the place is 3.0×10^{-5} T.

SUMMARY

In this session, you have learnt that:

1. The magnetic flux ϕ can be defined as a quantitative measure of the number of magnetic field lines **B** crossing a particular surface area **A** normally. It is defined as:

$$\phi = B \bullet A$$

2. Faraday's law of electromagnetic induction states that whenever the magnetic flux through a coil changes with time, an emf will be induced in the coil. The magnetic

flux can be changed by changing the magnetic field B, the area A of the loop, or the direction between the magnetic field and the area vector.

$$\varepsilon = -\frac{d\phi}{dt}$$

3. Lenz's law states that the direction of an induced emf is such that any current it produces, always opposes, through the magnetic field of the induced current, the change inducing the emf
4. In Mutual Induction, changing the magnetic flux in one coil induces an emf in an adjascent coil. The coefficient of mutual inductance M is defined as:

$$M = -\frac{\varepsilon_2}{\frac{d i_1}{dt}}$$

The induced emf in coil 1 caused by a changing current in coil 2 is given by:

$$\varepsilon_1 = -M\frac{d i_2}{dt}$$

5. In Self-Induction, changing the magnetic flux in a coil will induce an emf in that coil. The induced emf opposes the changing magnetic flux.

   The *self-inductance of the solenoid coil* designated by L is defined as:

$$L = \mu_0 A l n^2$$

   The self- induced emf is defined as:

$$\varepsilon = -L\frac{di}{dt}$$

6. An inductor is a circuit element in which a self-induced emf accompanies a changing current.
7. The energy stored in magnetic field of a solenoid is given by:

$$U = \frac{1}{2}\frac{B^2 A l}{\mu_0}$$

8. The energy stored in magnetic field of an inductor is given by:

$$W = \frac{1}{2}L I^2$$

9. Magnetic energy density or energy stored per unit volume that is stored in a magnetic field is:

$$u = \frac{1}{2}\frac{B^2}{\mu_0}$$

Study Session 12: Alternating Current Voltages Applied to Inductors, Capacitors and Resistors

Expected Duration: 1 week or 2 contact hours

Introduction

In session 11, you learnt that changing magnetic flux can induce an emf according to Faraday's law of induction. If a coil rotates in the presence of a magnetic field, the induced emf varies sinusoidally with time and leads to an alternating current (AC).Alternating Current or Voltage varies in magnitude and its polarity reverses periodically. It means that the alternating current will flow first in one direction and then in opposite direction. All appliances around us run on alternating current. Any appliance that you plug into a wall socket uses ac. In this session, you will learn how resistors, inductors and capacitors behave in circuits with sinusoidally varying voltages and currents.

Learning Outcomes

When you have studied this session, you should be able to explain the:

12.1 Alternating Currents and Voltages

12.2 Root-Mean-Square Value of an Alternating Current

12.3 A.C Resistor Circuit

12.4 A.C Capacitor Circuit

12.5 A.C Inductive Circuit

12.6 Resistor, Inductor and Capacitor in Series

12.1 Alternating Currents and Voltages

When a battery is connected to a resistor, charge flows through the resistor in one direction only. The direction of the current can be reversed by interchanging the battery connections. However, the magnitude of the current will remain constant. Such a current is called *direct current.* But a current whose magnitude changes continuously and direction changes periodically is said to be *an alternating current*. A source of alternating current (ac) is symbolized by a wavy line enclosed in a circle as shown in figure 12.1.

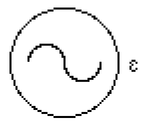


Figure 12.1. Symbol of AC source

The time dependence of the AC or the voltage of the AC source is of the form:

$$V = V_m \cos \omega t \qquad (12.1a)$$

$$I = I_m \cos \omega t \qquad (12.1b)$$

V_m and I_m are known as the peak values of the alternating current respectively.

12.2 Root-Mean-Square Value of an Alternating Current

The RMS or Root Mean Square value of alternating current is defined as that value of the steady current, which when passed through a resistor for a given time, will generate the same amount of heat as generated by an alternating current when passed through the same resistor for the same time.

The RMS value is also called effective value of an ac and is denoted by I_{rms} or I_{eff} .

When an alternating current $I = I_0 \sin \omega t$ flows through a resistor of resistance R, the amount of heat produced in the resistor in a small time *dt* is:

$$dH = i^2 R\,dt$$

The total amount of heat produced in the resistance in one complete cycle is:

$$H = \int_0^{\tau} i^2 R\,dt = \int_0^{\tau} I_0^2 \sin^2 \omega t\, R\,dt \qquad (12.2a)$$

$$H = \frac{I_0^2 RT}{2} \qquad (12.2b)$$

This heat is also equal to the heat produced by rms value of AC in the same resistor R and in the same time T.

$$H = I_{rms}^2 RT \qquad (12.3)$$

$$\therefore\ I_{rms}^2 RT = \frac{I_0^2 RT}{2} \ \Rightarrow\ I_{rms} = \frac{I_0}{\sqrt{2}} = 0.707 I_0 \qquad (12.4)$$

Similarly,

$$V_{rms} = \frac{V_0}{\sqrt{2}} = 0.707V_0 \qquad (12.5)$$

Thus, the rms value of an a.c is 0.707 times the peak value of the a.c.

12.3 A. C Circuit with a Resistor

Figure 12.2 shows a single-loop circuit with a source of alternating emf and a resistor. The current through the resistor is a function of time. The magnitude of this current can be obtained through Kirchhoff's second rule which implies that:

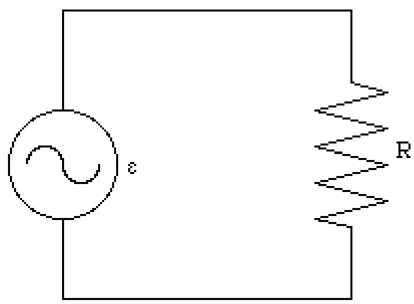


Figure 12.2 Single –loop A.C resistor circuit

$$\varepsilon(t) = -I(t)R = 0 \qquad (12.6a)$$

$$I(t) = \frac{\varepsilon(t)}{R} = \frac{\varepsilon_0 \sin \omega t}{R} \qquad (12.6b)$$

$$I(t) = I_0 \sin \omega t \qquad (12.6c)$$

where $I_0 = \frac{\varepsilon_0}{R}$ is the peak value of a.c in the circuit. Equation (12.6c) gives the instantaneous value of current in the circuit containing R. It shows that the current oscillates in phase with the emf.

12.4 A.C Circuit with a Capacitor

Figure 12.3 shows an a.c source ε generating a.c voltage $V = V_m \sin \omega t$ connected to a capacitor only, a purely capacitive a.c circuit. The charge on the capacitor at any time can be obtained by applying Kirchhoff's second rule to the circuit shown in figure 12.3 and is equal to

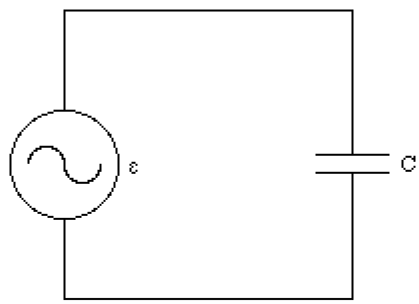


Figure 12.3 : A.C capacitor circuit

$$Q(t) = CV_m \sin \omega t \qquad (12.7)$$

The current in the circuit can be obtained by differentiating equation (12.7) with respect to time.

$$I(t) = \frac{dQ}{dt} = \omega C V_m \cos \omega t \qquad (12.8)$$

Equation (12.8) can be written as:

$$I(t) = \frac{V_m}{X_C} \cos \omega t \qquad (12.9)$$

Thus, $\frac{1}{\omega C} = X_C$ (12.10)

X_C is the resistance offered by the capacitor. It is called capacitive reactance measured in ohms. From equation (12.9), it follows that in an a.c circuit with a capacitor, the current leads the voltage by a phase angle of 90^0. In other words, the voltage lags behind the current by a phase angle of 90^0.

12.5 A.C Circuit with an Inductor

Figure 12.4 shows a circuit consisting of an inductor and a source of alternating emf. The self-induced emf across the inductor is $L\frac{dI}{dt}$. Applying Kirchhoff's second rule to the circuit shown in figure 12.4 gives:

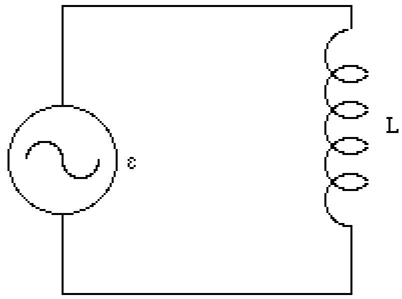


Figure 12.4: A.C Inductive Circuit

$$V_0 \sin \omega t - L\frac{dI}{dt} = 0 \qquad (12.11)$$

$$\frac{dI}{dt} = \frac{V_0}{L}\sin \omega t \;\Rightarrow\; I = \int \left(\frac{V_0}{L}\right)\sin \omega t \, dt \qquad (12.12)$$

$$I = \left(\frac{V_0}{\omega L}\right)(-\cos \omega t) \qquad (12.13)$$

But from trigonometry, $(-\cos \omega t) = \sin(\omega t - 90^0)$

Therefore, $I = I_0 \sin(\omega t - 90^0)$ (12.14)

where $I_0 = \dfrac{V_0}{L\omega}$. The equations for current and voltage show that the current through the inductor lags behind the voltage by 90^0.

The ratio of the peak value of voltage across the inductor to the peak value of current through it is constant and is called the inductive reactance, X_L, which is similar to the resistance.

$$X_L = \frac{V_0}{I_0} = L\omega = 2\pi f L \qquad (12.15)$$

Therefore, inductive reactance depends on the supply frequency, that is, X_L is directly proportional to the frequency. The unit of inductive reactance X_L is ohms. It can be concluded that for a purely inductive circuit, the current lags behind the voltage by 90^0.

12.6 Resistor, Inductor and Capacitor in Series

Let an alternating source of emf ε be connected to a series combination of a resistor of resistance R, inductor of inductance L and a capacitor of capacitance C as shown in figure 12.5.

Let the current flowing through the circuit be I.

The voltage across the inductor coil is $V_L = I X_L$

The voltage across the capacitor is $V_C = I X_C$

The voltage drop across the resistor is $V_R = I R$

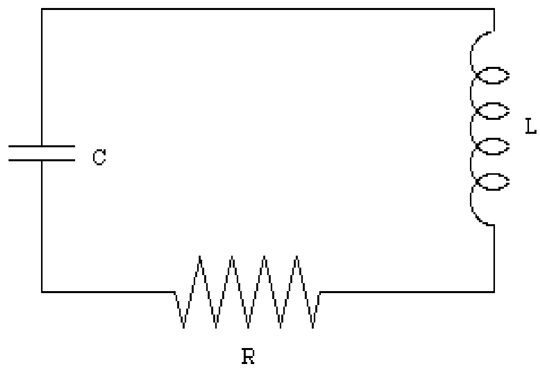


Figure 12.5a: LRC Circuit

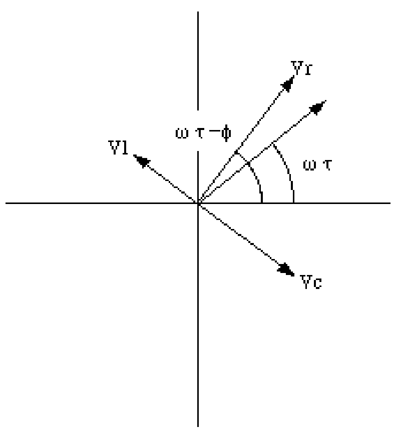


Figure 12.5 b: Phasor diagram for LCR circuit

The voltages across different components are represented in the voltage phasor diagram (Fig.12.5b). V_L and V_C are 180^0 out of phase with each other and the resultant of V_L and V_C is $(V_L - V_C)$. The applied voltage 'V' is equal to the vector sum of V_L, V_C and V_R.

$$V^2 = V_R^2 + (V_L - V_C)^2$$

$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$V = \sqrt{(I R)^2 + (I X_L - I X_C)^2}$$

$$V = I \sqrt{R^2 + (X_L - X_C)^2}$$

$$\frac{V}{I} = Z \sqrt{R^2 + (X_L - X_C)^2} \qquad (12.16)$$

The expression $\sqrt{R^2 + (X_L - X_C)^2}$ is the overall effective opposition offered by the combination of resistor, inductor and capacitor known as the impedance of the circuit. It is represented by Z and is measured in ohm. The phase angle ϕ between the voltage and current is given by:

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{I X_L - I X_C}{I R} \qquad (12.17)$$

$$\tan\phi = \frac{X_L - X_C}{R} \Rightarrow \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \qquad (12.18)$$

The value of current at any instant in a series RLC circuit is given by:

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad (12.19)$$

At a particular value of the angular frequency, the inductive reactance and the capacitive reactance will be equal to each other, $\omega L = \frac{1}{\omega C}$, so the impedance becomes minimum and it is given by Z = R.

The particular frequency f_0 at which the impedance of the circuit becomes minimum and therefore the current becomes maximum is called resonant frequency of the circuit. Such a circuit which admits maximum current is called series resonance circuit. Thus, the maximum current through the circuit at resonance is given by:

$$I_0 = \frac{V}{R} \qquad (12.20)$$

Maximum current flows through the circuit, since the impedance of the circuit is equal to the ohmic resistance of the circuit, i.e, Z = R

$$\omega L = \frac{1}{\omega C}$$

$$\omega = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \qquad (12.21)$$

The average power dissipated in an RLC circuit is given by:

$$P = I^2 Z \cos\phi \qquad (12.22)$$

The quantity $\cos\phi$ is called the power factor in the circuit.

Worked Examples

1. A series RLC circuit with R = 10.0 Ω, L = 400 mH and C = 2.0 μF is connected to an AC voltage source $V(t) = V_0 \sin\omega t$ which has a maximum amplitude $V_0 = 100V$.

(a) Determine the resonant frequency ω_0.

(b) Calculate the rms current at resonance

(c) Let the driving frequency $\omega = 4000\, rad\, s^{-1}$. Assume the current response is given by $I(t) = I_0 \sin(\omega t - \phi)$. Calculate the amplitude of the current and the phase shift between the current and the driving voltage.

Solution:

(a) $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(400\, mH)(2.0\, \mu F)}} = \frac{1}{\sqrt{8 \times 10^{-7}\, s}} = 1.1 \times 10^3\ rad\, s^{-1}$

(b) At resonance, Z = R . Therefore,

$$I_{rms} = \frac{V_{rms}}{R} = \frac{\left(\frac{V_0}{\sqrt{2}}\right)}{R} = \frac{\left(\frac{100V}{\sqrt{2}}\right)}{10.0\,\Omega} = 7.07\ A$$

(c) $X_C = \frac{1}{\omega C} = \frac{1}{(4000\, rad\, s^{-1})(2.0\, \mu F)} = 125\,\Omega$

$$X_L = \omega L = (4000\, rad\, s^{-1})(400\, mH) = 1600\,\Omega$$

$$\sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(10.0\,\Omega)^2 + (1600\,\Omega - 125\,\Omega)} = 1475\,\Omega$$

So $I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{100\,V}{\sqrt{(10.0\,\Omega)^2 + (1600\,\Omega - 125\,\Omega)^2}}$

$$I_0 = \frac{100V}{1475\,\Omega} = 6.8 \times 10^{-2}\ A$$

$$\tan\phi = \frac{X_L - X_C}{R} \Rightarrow \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{1600\,\Omega - 125\,\Omega}{10.0\,\Omega}\right) = 89.6^0$$

2. An alternating voltage is given by $V = 282.8\sin 314t\ Volts$. Determine the:
   (a) Rms voltage, (b) frequency and (c) instantaneous value of voltage when $t = 4ms$

   Solution:

   (a) The general expression for an alternating voltage is:

   $$V = V_m \sin(\omega t \pm \phi)$$

   Comparing $V = 282.8\sin 314t\ Volts$ with this general expression gives the peak voltage as 282.8 V.

   Hence the rms voltage $= 0.707 \times$ maximum value of voltage

   $$= 0.707 \times 282.7\,V = 200\ V$$

   (b) Angular velocity $\omega = 314\ rad\,s^{-1} \Rightarrow 2\pi f = 314$

   Hence, frequency $f = \dfrac{314}{2\pi} = 50\ Hz$

   (c) When $t = 4\,ms$ $V = 282.8\sin(314 \times 4 \times 10^{-3}) = 282.8\ \sin(1.256) = 268.9\,V$

3. In an oscillating LC circuit, L = 1.10 H and C = 4.00 µF. The maximum charge on the capacitor is 3.00 µC. Calculate the maximum current.

   Solution:

   The circuit oscillates with $q(t) = q_{max}\cos\omega t$

   The current $i = \dfrac{dq}{dt} = -q_{max}\,\omega\sin\omega t \quad \Rightarrow i_{max} = -\omega q_{max}$

   $$|i_{max}| = \omega q_{max} = q_{max}\cdot\frac{1}{\sqrt{LC}} = \frac{3\times10^{-6}}{\sqrt{(1.10\times10^{-3})(4.00\times10^{-6})}} = 4.52\times10^{-2}\ A$$

4. What direct current will produce the same amount of thermal energy in a particular resistor, as an alternating current that has a maximum value of 2.60 A?

   Solution:

   In an a.c , $i = i_{max}\sin\omega t$

   $$P_{av} = \frac{i_{max}^2 R}{2}$$

   The direct current I_{DC} is given by:

   $$P_{av} = I_{DC}^2 R \Rightarrow I_{DC} = \frac{I_{max}}{\sqrt{2}} = \frac{2.60\ A}{\sqrt{2}} = 1.84\,A$$

Self – Assessment Questions (SAQs)

1. The current in an a.c circuit at any time *t* seconds is given by:

   $I = 120 \sin(100\pi t + 0.36)$ amperes. Determine the:

(a) Peak value, periodic time, frequency and phase angle relative to $120 \sin 100\pi t$
(b) Value of the current when t = 8 ms
(c) Time when the current first reaches 60A

2. Determine the maximum value of an a.c voltage whose rms value is 100 V.
3. An a.c generator has emf $\varepsilon = \varepsilon_m \sin \omega_d t$, with $\varepsilon_m = 25.0\,V$ and $\omega_d = 377\ rad\,s^{-1}$. It is connected to a 12.7 H inductor.
   (a) What is the maximum value of the current?
   (b) When the current is a maximum, what is the emf of the generator?
4. In an oscillating LC circuit with L = 50 mH and C = 4.0 μF, the current is initially a maximum. How long will it take before the capacitor is fully charged for the first time?
5. An oscillating LC circuit consists of a 75.0 mH inductor and a 3.60 μF capacitor. If the maximum charge on the capacitor is 2.90 μC. Calculate the:
   (a) Total energy in the circuit.
   (b) Maximum current
6. At what frequency would a 6.0 mH inductor and a 10 μF capacitor have the same reactance?

SUMMARY

In this session, you have learnt that:

1 In an a.c circuit, the voltage across the source is given by $V = V_m \cos \omega t$ and current $I = I_m \cos \omega t$ where V_m and I_m are known as the peak values of the alternating current respectively.

2. The RMS or Root Mean Square value of alternating current is defined as that value of the steady current, which when passed through a resistor for a given time, will generate the same amount of heat as generated by an alternating current when passed through the same resistor for the same time.

3. The particular frequency f_0 at which the impedance of the circuit becomes minimum and therefore the current becomes maximum is called resonant frequency of the circuit.

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

4. In a purely capacitive a.c circuit with a capacitor, the current leads the voltage by a phase angle of 90^0 . X_C is the resistance offered by the capacitor. It is called capacitive reactance measured in ohms.

$$\frac{1}{\omega C} = X_C$$

5. The ratio of the peak value of voltage across the inductor to the peak value of current through it is constant and is called the inductive reactance, X_L. In a purely inductive a.c circuit, the current lags the voltage by 90^0.

$$X_L = \frac{V_0}{I_0} = L\omega = 2\pi f L$$

6.In a series LCR circuit, the current is given by:

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$