COURSE MANUAL

Statistical Inference I

STA 121

University of Ibadan Distance Learning Centre
Open and Distance Learning Course Series Development

ISBN: 978-021-274-4

General Editor: Prof. Bayo Okunade

University of Ibadan Distance Learning Centre
University of Ibadan,
Nigeria

Telex: 31128NG

Tel: +234 (80775935727)
E-mail: ssu@dlc.ui.edu.ng
Website: www.dlc.ui.edu.ng

Vice-Chancellor’s Message

The Distance Learning Centre is building on a solid tradition of over two decades of service in the provision of External Studies Programme and now Distance Learning Education in Nigeria and beyond. The Distance Learning mode to which we are committed is providing access to many deserving Nigerians in having access to higher education especially those who by the nature of their engagement do not have the luxury of full time education. Recently, it is contributing in no small measure to providing places for teeming Nigerian youths who for one reason or the other could not get admission into the conventional universities.

These course materials have been written by writers specially trained in ODL course delivery. The writers have made great efforts to provide up to date information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly.

In addition to provision of course materials in print and e-format, a lot of Information Technology input has also gone into the deployment of course materials. Most of them can be downloaded from the DLC website and are available in audio format which you can also download into your mobile phones, IPod, MP3 among other devices to allow you listen to the audio study sessions. Some of the study session materials have been scripted and are being broadcast on the university’s Diamond Radio FM 101.1, while others have been delivered and captured in audio-visual format in a classroom environment for use by our students. Detailed information on availability and access is available on the website. We will continue in our efforts to provide and review course materials for our courses.

However, for you to take advantage of these formats, you will need to improve on your I.T. skills and develop requisite distance learning Culture. It is well known that, for efficient and effective provision of Distance learning education, availability of appropriate and relevant course materials is a *sine qua non*. So also, is the availability of multiple plat form for the convenience of our students. It is in fulfilment of this, that series of course materials are being written to enable our students study at their own pace and convenience.

It is our hope that you will put these course materials to the best use.

Prof. Abel Idowu Olayinka
Vice-Chancellor

Foreword

As part of its vision of providing education for “Liberty and Development” for Nigerians and the International Community, the University of Ibadan, Distance Learning Centre has recently embarked on a vigorous repositioning agenda which aimed at embracing a holistic and all encompassing approach to the delivery of its Open Distance Learning (ODL) programmes. Thus we are committed to global best practices in distance learning provision. Apart from providing an efficient administrative and academic support for our students, we are committed to providing educational resource materials for the use of our students. We are convinced that, without an up-to-date, learner-friendly and distance learning compliant course materials, there cannot be any basis to lay claim to being a provider of distance learning education. Indeed, availability of appropriate course materials in multiple formats is the hub of any distance learning provision worldwide.

In view of the above, we are vigorously pursuing as a matter of priority, the provision of credible, learner-friendly and interactive course materials for all our courses. We commissioned the authoring of, and review of course materials to teams of experts and their outputs were subjected to rigorous peer review to ensure standard. The approach not only emphasizes cognitive knowledge, but also skills and humane values which are at the core of education, even in an ICT age.

The development of the materials which is on-going also had input from experienced editors and illustrators who have ensured that they are accurate, current and learner-friendly. They are specially written with distance learners in mind. This is very important because, distance learning involves non-residential students who can often feel isolated from the community of learners.

It is important to note that, for a distance learner to excel there is the need to source and read relevant materials apart from this course material. Therefore, adequate supplementary reading materials as well as other information sources are suggested in the course materials.

Apart from the responsibility for you to read this course material with others, you are also advised to seek assistance from your course facilitators especially academic advisors during your study even before the interactive session which is by design for revision. Your academic advisors will assist you using convenient technology including Google Hang Out, You Tube, Talk Fusion, etc. but you have to take advantage of these. It is also going to be of immense advantage if you complete assignments as at when due so as to have necessary feedbacks as a guide.

The implication of the above is that, a distance learner has a responsibility to develop requisite distance learning culture which includes diligent and disciplined self-study, seeking available administrative and academic support and acquisition of basic information technology skills. This is why you are encouraged to develop your computer skills by availing yourself the opportunity of training that the Centre’s provide and put these into use.

In conclusion, it is envisaged that the course materials would also be useful for the regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks. We are therefore, delighted to present these titles to both our distance learning students and the university's regular students. We are confident that the materials will be an invaluable resource to all.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.

Professor Bayo Okunade
Director

Course Development Team

Content Authoring	Ojo, J.F Femi J. Ayoola
Content Editor	Prof. Remi Raji-Oyelade
Production Editor	Ogundele Olumuyiwa Caleb
Learning Design/Assessment Authoring	SkulPortal Technology
Managing Editor	Ogunmefun Oladele Abiodun
General Editor	Prof. Bayo Okunade

Table of Contents

Course Introduction

The course covers the basic knowledge of statistical inference, which is the act of making deductive statement about the related population from the quantity obtained from its representative sample. This is carried out through estimation and test of hypothesis. Other aspects considered are regression and correlation analysis as well as elementary time series analysis.

Objectives of the Course

The objectives of this course are to:

1. introduce you to statistical inference, which is a major division in the study of statistics as a course;
2. explain the basic principles/theories in statistical inference and its application in everyday usage; and
3. discuss basic statistical methods such as Regression and Correlation analysis as well as time series analysis.

At the end of the course, the you should:

1. have a working knowledge of statistical inference and its practical application in handling real life situation; and
    2. see regression and correlation as well as time series analysis as a course of study and as an applied discipline, which is used in solving real life problems.

Study Session 1: Introduction to Statistical Inference

Introduction

Statistical inference means drawing conclusions based on data. There are a many contexts in which inference is desirable, and there are many approaches to performing inference. One important inferential context is parametric models. For example, if you have noisy (x; y) data that you think follow the pattern y = 0 + 1x + error, then you might want to estimate 0, 1, and the magnitude of the error.

The aim of this study session is to introduce you to the meaning of statistical inference. You shall commence by giving the meaning and definition of statistics. Furthermore, you shall also examine the components of statistical inference. The other branches of statistics, apart from statistical inference, will also be treated.

Learning Outcomes for Study Session for Study Session 1

At the end of this study session, you should be able to:

1.1 Definition of Statistics

1.2 Explain Types of Statistics

1.3 Discuss on Population

1.1 Definitions of Statistics

Branch of mathematics concerned with collection, classification, analysis, and interpretation of numerical facts, for drawing inferences on the basis of their quantifiable likelihood (probability). Statistics can interpret aggregates of data too large to be intelligible by ordinary observation because such data (unlike individual quantities) tend to behave in regular, predictable manner.

1.1.1 History of Statistics

The origin of statistics may be traced to two areas of interest, which are very dissimilar: games of chance and what we now call political science. In the middle of eighteenth century, studies in probability (motivated in part by interest in games of chance) led to the mathematical treatment of errors of measurement and the theory, which now forms the foundation of statistics.

In the same century, interest in the description and analysis of political units led to the development of methods, which have now come under the heading of descriptive statistics.

Statistics is a relatively new subject in the curricula of institutions of learning, but its use is as old as history itself. The word ‘statistics’ originated from the Latin word “Status’ that is “State”.

Its formalization as a teaching and practising discipline emanated from a realization of its indispensability in decision-making processes in all areas of human endeavours. In everyday life, we encounter problems that need scientific decisions or solution. Many of these decisions may be quite simple, while some are so difficult that they need to be studied with relevant information.

For instance, a lecturer may be late in arriving for his lectures; students are faced with the decision to wait or not to wait for him.

1. If the lecturer does not usually come late for his lectures, you may decide to go away.
2. If he usually comes late, then you may decide to wait for him.

In both cases the students base their decision on a degree of rational belief called probability. Decisions in social and management sciences, biological sciences, technology and medical sciences to mention a few require quantitative information, and its analysis facilitates action by ensuring our understanding of the mechanics of the underlying phenomena.

In its initial conception, statistics is the collection of the population and of socio-economic information vital to the state. The state requires information on the number of taxable adults, to allow for a projection of reliable total income. In this century, statistics has advanced far beyond that narrower conception.

In general, the word “statistics” has three possible connotations, depending on the context of the application: as a subject, as a piece of information and as a mathematical function. Statistics, as a subject is defined as the scientific methodology which is concerned with the planning, collection, summarization, presentation, analyzing and interpretation of data to meet a lot of specified objectives.

As a piece of information, statistics is a piece of quantitative data such as graduates’ data, export data, etc. As a mathematical entity, statistics is a function of observation. In a nutshell, statistics can be defined as the study of the techniques and theory involved in the planning, collection, summarizing, analyzing and interpretation of data and the subsequent utilization of the results.

In Text Question

Branch of mathematics concerned with collection, classification, analysis, and interpretation of numerical facts, for drawing inferences on the basis of their quantifiable likelihood is called ___

(a) Strategy

(b) Statistics

(c) Inference

(d) Numerical

In Text Answer

The answers is (b) Statistics

1.2 Types of Statistics

Inferential Statistics. Inferential statistics, as the name suggests, involves drawing the right conclusions from the statistical analysis that has been performed using descriptive statistics. In the end, it is the inferences that make studies important and this aspect is dealt with in inferential statistics.

Statistics can be divided into three parts namely:

- **Descriptive Statistics**

This has to do with the reduction of mass of data into few members, which are quantitative expressions of the salient characteristics of the data. Also it summarizes and gives a descriptive account of numerical information in form of reports, charts and diagrams. Examples include all measures of central tendency, variation and partition and presentation in tabular and diagrammatic forms.

It also Deals with the presentation and collection of data. This is usually the first part of a statistical analysis. It is usually not as simple as it sounds, and the statistician needs to be aware of designing experiments, choosing the right focus group and avoid biases that are so easy to creep into the experiment.

Different areas of study require different kinds of analysis using descriptive statistics. For example, a physicist studying turbulence in the laboratory needs the average quantities that vary over small intervals of time. The nature of this problem requires that physical quantities be averaged from a host of data collected through the experiment.

- **Statistical Method**

Statistical method is a device for classifying data and making clear relationship existing between them as well as the use of statistical tools to bring out the salient points.

Mathematical concepts, formulas, models, techniques used in statistical analysis of random data. In comparison, deterministic methods are used where the data is easily reproducible or where its behavior is determined entirely by its initial stage and inputs.

- **Statistical Inference**

Although descriptive statistics is an important branch of statistics and it continues to be widely used, statistical information usually arises from samples (from observations made on only part of a large set of items), and this means that its analysis will require generalizations which go beyond the data.

As a result, the most important feature of the recent growth of statistics has been a shift in emphasis from methods, which merely describe to methods, which serve to make generalizations; that is, a shift in emphasis from descriptive statistics to the methods of statistical inference, or inductive statistics.

Since most of our everyday problems consist of decision making, and for a statistician a decision is usually about a statistical population on the basis of evidence collected from a sample taken from that population, the state of the population shall be called the state of nature. It is the true state of things.

We may, by taking planned observation, be able to make some inference about the population. If the state of nature is represented by a probability model, the inference made about the state of nature on the evidence provided by the sample is called statistical inference. In other words, statistical inference is the process of drawing inference about the population from the sample.

In Text Question

There are three types of statistics they are Descriptive Statistics, statistical method and statistics inference. **True/False**

In Text Answer

True

1.3 Population

Population is a collection of the individual items, whether of people or thing, that are to be observed in a given problem situation. You can also use the word to describe a collection of

1. Human beings;
2. Animals, e.g. goats, cattle, birds and rats;
3. Inanimate objects, e.g. chairs, tables and farms;
4. Even a part of a given population like a class of students listening to a statistics lecture.

A population can be finite or infinite, countable or uncountable. Closely related to a population is a sample. Quite often, we cannot reach all the units of a population even if we have all the resources in the world to do so. Whether we like it or not, we have to be satisfied with a sample.

1. **Finite Population:** A population is said to be finite if it consists of a finite or fixed number of elements (items, objects, and measurements or observations)
2. **Infinite Population:** A population is said to be infinite if there is (at least hypothetically) no limit to the number of elements it can contain. For example, a possible roll of a pair of dice is an infinite population for there is no limit to the number of times they can be rolled.

Example 1

The population for a study of infant health might be all the children born in Nigeria in the 1980s. The sample might be all babies born on 7th May in any of the years.

1.3.1 Sample

A sample is a group of units selected from a larger group (the population). By studying the sample, it is hoped to draw valid conclusions about the larger group. A sample is generally selected for study because the population is too large to study in its entirety. The sample should be representative of the general population.

This is often best achieved by random sampling. Also, before collecting the sample, it is important that the researcher carefully and completely defines the population, including a description of the members to be included.

1.3.2 Advantages of sampling

Sampling ensures convenience, collection of intensive and exhaustive data, suitability in limited resources and better rapport. In addition to this, sampling has the following advantages also.

1. Low cost of sampling

If data were to be collected for the entire population, the cost will be quite high. A sample is a small proportion of a population. So, the cost will be lower if data is collected for a sample of population which is a big advantage.

2. Less time consuming in sampling

Use of sampling takes less time also. It consumes less time than census technique. Tabulation, analysis etc., take much less time in the case of a sample than in the case of a population.

3. Scope of sampling is high

The investigator is concerned with the generalization of data. To study a whole population in order to arrive at generalizations would be impractical. Some populations are so large that their characteristics could not be measured. Before the measurement has been completed, the population would have changed.

But the process of sampling makes it possible to arrive at generalizations by studying the variables within a relatively small proportion of the population.

4. Accuracy of data is high

Having drawn a sample and computed the desired descriptive statistics, it is possible to determine the stability of the obtained sample value.

A sample represents the population from which its is drawn. It permits a high degree of accuracy due to a limited area of operations.

Moreover, careful execution of field work is possible. Ultimately, the results of sampling studies turn out to be sufficiently accurate.

5. Organization of convenience

Organizational problems involved in sampling are very few. Since sample is of a small size, vast facilities are not required. Sampling is therefore economical in respect of resources. Study of samples involves less space and equipment.

6. Intensive and exhaustive data

In sample studies, measurements or observations are made of a limited number. So, intensive and exhaustive data are collected.

7. Suitable in limited resources

The resources available within an organization may be limited. Studying the entire universe is not viable. The population can be satisfactorily covered through sampling. Where limited resources exist, use of sampling is an appropriate strategy while conducting marketing research.

8. Better rapport

An effective research study requires a good rapport between the researcher and the respondents. When the population of the study is large, the problem of rapport arises. But manageable samples permit the researcher to establish adequate rapport with the respondents.

1.3.3 Disadvantages of sampling

The reliability of the sample depends upon the appropriateness of the sampling method used. The purpose of sampling theory is to make sampling more efficient. But the real difficulties lie in selection, estimation and administration of samples.

1. Chances of bias

The serious limitation of the sampling method is that it involves biased selection and thereby leads us to draw erroneous conclusions. Bias arises when the method of selection of sample employed is faulty. Relative small samples properly selected may be much more reliable than large samples poorly selected.

2. Difficulties in selecting a truly representative sample

Difficulties in selecting a truly representative sample produce reliable and accurate results only when they are representative of the whole group. Selection of a truly representative sample is difficult when the phenomena under study are of a complex nature. Selecting good samples is difficult.

3. Inadequate knowledge in the subject

Use of sampling method requires adequate subject specific knowledge in sampling technique. Sampling involves statistical analysis and calculation of probable error. When the researcher lacks specialized knowledge in sampling, he may commit serious mistakes. Consequently, the results of the study will be misleading.

4. Changeability of units

When the units of the population are not in homogeneous, the sampling technique will be unscientific. In sampling, though the number of cases is small, it is not always easy to stick to the, selected cases. The units of sample may be widely dispersed.

Some of the cases of sample may not cooperate with the researcher and some others may be inaccessible. Because of these problems, all the cases may not be taken up. The selected cases may have to be replaced by other cases. Changeability of units stands in the way of results of the study.

5. Impossibility of sampling

Deriving a representative sample is di6icult, when the universe is too small or too heterogeneous. In this case, census study is the only alternative. Moreover, in studies requiring a very high standard of accuracy, the sampling method may be unsuitable. There will be chances of errors even if samples are drawn most carefully.

Summary

In this study session you have learnt about:

1. **Statistics is a** Branch of mathematics concerned with collection, classification, analysis, and interpretation of numerical facts, for drawing inferences on the basis of their quantifiable likelihood.
2. **Types of Statistics**

   Inferential Statistics. Inferential statistics, as the name suggests, involves drawing the right conclusions from the statistical analysis that has been performed using descriptive statistics.

Statistics can be divided into three parts namely:

- Descriptive Statistics
- Statistical Method
- Statistical Inference

3. **Population**

Population is a collection of the individual items, whether of people or thing, that are to be observed in a given problem situation.

Self-Assessment Questions (SAQs) for study session 1

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 1.1 (Testing Learning Outcomes 1.1)

Define Statistics

SAQ 1.2 (Testing Learning Outcomes 1.2)

Discuss Type of Statistics

SAQ 1.3 (Testing Learning Outcomes 1.3)

Explain Population

References

Adamu, S. O. and Johnson, T. L.(1997) *Statistics for Beginners.* Ibadan: Book I SAAL Publications

John, E. F.(1974). *Modern Elementary Statistics*, London: International Edition. Prentice Hall.

Murray, R. S. (1972) *Schaum's Outline Series. Theory and Problems of Statistics*. New York: McGraw-Hill Book Company.

Olubusoye O. E. et all (2002) *Statistics for Engineering, Physical and Biological Sciences*. Ibadan: A Divine Touch Publication

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). *Statistical Theory and Methods* Ibadan: Joytal. Press.

Study Session 2: Elementary Idea of Sampling

Introduction

Sampling is concerned with the selection of a subset of individuals from within a statistical population to estimate characteristics of the whole population. Each observation measures one or more properties (such as weight, location, color) of observable bodies distinguished as independent objects or individuals.

The aim of this study session is to introduce you to the idea of sampling. Probability and non-probability sample will be discussed. Sampling with and without replacement will be highlighted. Sampling distribution will be discussed. Finally, we shall work examples on sampling distributions.

Learning Outcomes for Study Session 2

At the end of this study session, you should be able to:

2.1 Explain the Sampling

2.2 Discuss Non-Probability Sampling Techniques

2.3 Highlight on Sampling Concept

2.1 Sampling Techniques

Sampling is a method of selecting a subset or part of a population that is representative of the entire population. Various sampling designs and techniques have been developed in an attempt to improve this representation.

There are two major categories:

Probability \Random Sampling
Non-Probability Sampling

Figure 2.1: Major Categories

A sample is called a probability sample or random sample if each elementary unit or member of the population is being included in the sample. Some of the probability sampling techniques are:

1. **Simple random sampling:** This is one where each member of the population has equal chances of being included in the sample. This is achieved by use of the
    a. Lottery method, or
    b. Random number table.
2. **Systematic Sampling:** This is one where sample elements are selected at regular intervals from the sampling frame; the list of all the elements in the population.
3. **Stratified Sampling:** This is a technique where the population is divided into groups called strata and sample elements are selected using the simple random sampling from each group. Each group should be as homogeneous as possible.
4. **Cluster Sampling:** Here the population is divided into groups and a random sample of groups is selected. All the elements in the selected groups are investigated. It is commonly used, when there is no adequate list of elementary units called frame.
5. **Multistage Sampling**: This is one where, population elements are selected in two or more stages. For example, in a two - stage sample, the whole country is divided into state and local governments. The first stage could be a random selection of five states

and the second stage could be the selection of local governments from the selected states.

In Text Question

This is a technique where the population is divided into groups called _____

(a) Cluster Sampling

(b) Multistage Sampling

(c) Stratified

(d) Simple Random

In Text Answer

The answer is (c) stratified

2.2 Non-Probability Sampling Techniques

Some of the Non-probability sampling techniques are:

1. **Quota Sampling:** This is one where, although the population is divided into identified groups, elements are selected from each group without recourse to randomness. Here the interviewer is free to use his discretion to select the units to be included in the sample. This method is commonly used in opinion poll, by the journalist and in market research.
2. **Judgmental or Purposive Sampling:** This is a sample whose elementary units are chosen according to the discretion of expert who is familiar with the relevant characteristics of the population. These sampling units are selected judgmentally, and there is a heavy possibility of biasness.

2.2.1 Sampling with and without Replacement

The sampling method, where each member of a population may be chosen more than once is called sampling with replacement. However, if each member cannot be chosen more than once, such a sampling method is called sampling without replacement.

Explain 1

To illustrate the notion of a random sample from a finite population, let us consider first a finite population, consisting of 5 elements which we shall label a, b, c, d, e. This might be the incomes of 5 professors, weights of 5 students and so on. To begin with, let us see how many different samples of say, size 3, can be taken from this finite population.

Answer: There are ${}^{n}C_{r}=\frac{n!}{r!(n-r)!}$ ways in which r objects can be selected from a set of n objects. n = 5, and r = 3, ${}^{5}C_{3}$ = 10 different samples namely abc, abd, abe, bcd, bce, cde, acd, ace, ade, bde. If we select one of the 10 possible samples in such a way that each has the same probability of being chosen, we say that we have a simple random sample.

Explain 2

Suppose we have a bowl of 100 unique numbers from 0 to 99. We want to select a random sample of numbers from the bowl. After we pick a number from the bowl, we can put the number aside or we can put it back into the bowl. If we put the number back in the bowl, it may be selected more than once; if we put it aside, it can selected only one time.

When a population element can be selected more than one time, we are sampling with replacement. When a population element can be selected only one time, we are sampling without replacement.

2.2.3 Sampling Distributions

Sampling distribution concept ties in closely with the idea of chance variation or chance fluctuations for measuring the variability of data. For random samples of size n from a population having the mean μ and the standard deviation σ, the sampling distribution of $\overline{x}$ has the mean $\mu_{\overline{x}}=\mu$ and its standard deviation is given by $\sigma_{\overline{X}}=\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$ if sampling is without replacement, and $\sigma_{\overline{x}}=\frac{\sigma}{\sqrt{n}}$ if sampling is with replacement. It is customary to refer to $\sigma_{\overline{x}}$, the standard deviation of the sampling distribution of the mean, as the standard error of the mean. Its role in statistics is fundamental, as it measures the extent to which means fluctuate, or vary, due to chance.

Example

Let the ages at last birthday of 5 children be 2, 3, 6, 8, and 11, suppose that two of the children are selected at random with replacement. Calculate the following:

1. Mean and Standard deviation
2. List the possible sample of size 2
3. Using the result of part 2, construct a sampling distribution of the mean for random

samples of size two.

4. Calculate the mean and standard deviations of the probability distribution obtained in part 3 and verify the result with the result in 1.

Solution

1. $$\mu = \frac{\sum_{i=1}^{5} X_i}{N}$$

$$= \frac{2+3+6+8+11}{5} = 6$$

$$\sigma^2 = \frac{\sum_{i=1}^{5} (\mathrm{X}_i - \mu)^2}{N}$$

$$= \frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5} = 10.8$$

σ = 3.29

2. There are 5(5) = 25 samples of size two which can be drawn with replacement as follows:

(2, 2)	(2, 3)	(2, 6)	(2, 8)	(2, 11)
(3, 2)	(3, 3)	(3, 6)	(3, 8)	(3, 11)
(6, 2)	(6, 3)	(6, 6)	(6, 8)	(6, 11)
(8, 2)	(8, 3)	(8, 6)	(8, 8)	(8, 11)
(11, 2)	(11, 3)	(11, 6)	(11, 8)	(11, 11)

3.

S/N	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\bar{x}$	2	2.5	3	4	4.5	5	5.5	6	6.5	7	8	8.5	9.5	11
Probability	1/25	2/25	1/25	2/25	2/25	2/25	2/25	1/25	2/25	4/25	1/25	2/25	2/25	1/25

4.

$$\mu_{\bar{x}} = (2\times 1/25) + (2.5\times 2/25) + (3.0\times 1/25) + (4.0\times 2/25) + (4.5\times 2/25) + (5.0\times 2/25) + ... + (11\times 2/25)$$

=150/25 = 6.0

Illustrating the fact that $\mu_{\bar{x}} = \mu$

$$\sigma_{\bar{x}}^2 = (2.0-6.0)^2 \times \frac{1}{25} + ... + (11-6.0)^2 \times \frac{1}{25}$$

$$\sigma_{\bar{x}}^2 = \frac{135}{25} = 5.40$$

so that

$$\sigma_{\bar{x}} = \sqrt{5.40} = 2.32$$

This illustrates the fact that for sampling with replacement $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{10.8}{2} = 5.40$, agreeing with the above.

2.3 Sampling Concept

The following are the sampling concept:

a) Population

The collection of all units of a specified type in a given region at a particular point or period of time is termed as a population or universe. Thus, we may consider a population of persons, families, farms, cattle in a region or a population of trees or birds in a forest or a population of fish in a tank etc. depending on the nature of data required.

b) Sampling Unit

Elementary units or group of such units which besides being clearly defined, identifiable and observable, are convenient for purpose of sampling are called sampling units. For instance, in a family budget enquiry, usually a family is considered as the sampling unit since it is found to be convenient for sampling and for ascertaining the required information. In a crop survey, a farm or a group of farms owned or operated by a household may be considered as the sampling unit.

c) Sampling Frame

A list of all the sampling units belonging to the population to be studied with their identification particulars or a map showing the boundaries of the sampling units is known as sampling frame. Examples of a frame are a list of farms and a list of suitable area segments like villages in India or counties in the United States. The frame should be up to date and free from errors of omission and duplication of sampling units.

d) Random Sample

One or more sampling units selected from a population according to some specified procedures are said to constitute a sample. The sample will be considered as random or probability sample, if its selection is governed by ascertainable laws of chance.

In other words, a random or probability sample is a sample drawn in such a manner that each unit in the population has a predetermined probability of selection. For example, if a population consists of the N sampling units U1, U2,…,Ui,…,UN then we may select a sample of n units by selecting them unit by unit with equal probability for every unit at each draw with or without replacing the sampling units selected in the previous draws.

e) Non-random sample

A sample selected by a non-random process is termed as non-random sample. A Non-random sample, which is drawn using certain amount of judgment with a view to getting a representative sample is termed as judgment or purposive sample. In purposive sampling units are selected by considering the available auxiliary information more or less subjectively with a view to ensuring a reflection of the population in the sample.

This type of sampling is seldom used in large-scale surveys mainly because it is not generally possible to get strictly valid estimates of the population parameters under consideration and of their sampling errors due to the risk of bias in subjective selection and the lack of information on the probabilities of selection of the units.

f) Population parameters

Suppose a finite population consists of the N units U1, U2,…,UN and let Yi be the value of the variable y, the characteristic under study, for the ith unit Ui, (i=1,2,…,N). For instance, the unit may be a farm and the characteristic under study may be the area under a particular crop.

Any function of the values of all the population units (or of all the observations constituting a population) is known as a population parameter or simply a parameter. Some of the important parameters usually required to be estimated in surveys are population total $Y = \sum_{i-1}^{N} Y_i$ and population mean $\overline{Y} = \sum_{i-1}^{N} Y_i / N$

g) Statistic, Estimator and Estimate

Suppose a sample of n units is selected from a population of N units according to some probability scheme and let the sample observations be denoted by y1,y2,…,yn. Any function of these values which is free from unknown population parameters is called a statistic.

An estimator is a statistic obtained by a specified procedure for estimating a population parameter. The estimator is a random variable and its value differs from sample to sample and the samples are selected with specified probabilities. The particular value, which the estimator takes for a given sample, is known as an estimate.

h) Sample design

A clear specification of all possible samples of a given type with their corresponding probabilities is said to constitute a sample design.

For example, suppose we select a sample of n units with equal probability with replacement, the sample design consists of N possible samples (taking into account the orders of selection and repetitions of units in the sample) with 1/Nn as the probability of selection for each of them, since in each of the n draws any one of the N units may get selected. Similarly, in sampling n units with equal probability without replacement, the number of possible samples (ignoring orders of selection of units) is $\binom{N}{n}$ and the probability of selecting each of the samples is $1 \Big/ \binom{N}{n}$.

i) Unbiased Estimator

Let the probability of getting the i-th sample be Pi and let ti be the estimate, that is, the value of an estimator t of the population parameter θ based on this sample (i=1,2,…,Mo), Mo being the total number of possible samples for the specified probability scheme. The expected value or the average of the estimator t is given by $E(t)=\sum_{i=1}^{M_o} t_i P_i$

An estimator t is said to be an unbiased estimator of the population parameter θ if its expected value is equal to θ irrespective of the y-values. In case expected value of the estimator is not equal to population parameter, the estimator t is said to be a biased estimator of θ. The estimator t is said to be positively or negatively biased for population parameter according as the value of the bias is positive or negative.

j) Measures of error

Since a sample design usually gives rise to different samples, the estimates based on the sample observations will, in general, differ from sample to sample and also from the value of the parameter under consideration.

The difference between the estimate ti based on the i-th sample and the parameter, namely (ti - θ), may be called the error of the estimate and this error varies from sample to sample. An average measure of the divergence of the different estimates from the true value is given by the expected value of the squared error, which is $M(t)=E(t-\theta)^2=\sum_{i=1}^{M_o}(t_i-\theta)^2 P_i$ and this is known as mean square error (MSE) of the estimator. The MSE may be considered to be a measure of the accuracy with which the estimator t estimates the parameter.

The expected value of the squared deviation of the estimator from its expected value is termed sampling variance. It is a measure of the divergence of the estimator from its expected value and is given by

$$V(t)=\sigma^2 t=E\{t-E(t)\}^2=E(t)^2-\{E(t)\}^2$$

This measure of variability may be termed as the precision of the estimator t. The MSE of t can be expressed as the sum of the sampling variance and the square of the bias. In case of unbiased estimator, the MSE and the sampling variance are same.

The square root of the sampling variance $\sigma(t)$ is termed as the standard error (SE) of the estimator t. In practice, the actual value of $\sigma(t)$ is not generally known and hence it is usually estimated from the sample itself.

k) Confidence interval

The frequency distribution of the samples according to the values of the estimator t based on the sample estimates is termed as the sampling distribution of the estimator t. It is important to mention that though the population distribution may not be normal, the sampling distribution of the estimator t is usually close to normal, provided the sample size is sufficiently large.

If the estimator t is unbiased and is normally distributed, the interval $\{t \pm KSE(t)\}$ is expected to include the parameter θ in P% of the cases where P is the proportion of the area between –K and +K of the distribution of standard normal variate. The interval considered is said to be a confidence interval for the parameter θ with a confidence coefficient of P% with the confidence limit t – K SE(t) and t + K SE(t).

For example, if a random sample of the records of batteries in routine use in a large factory shows an average life t = 394 days, with a standard error SE(t) = 4.6 days, the chances are 99 in 100 that the average life in the population of batteries lies between

tL = 394 - (2.58)(4.6) = 382 days

tU = 394 + (2.58)(4.6) = 406 days

The limits, 382 days and 406 days are called lower and upper confidence limits of 99% confidence interval for t. With a single estimate from a single survey, the statement "θ lies between 382 and 406 days" is not certain to be correct. The "99% confidence" figure implies that if the same sampling plan were used may times in a population, a confidence statement being made from each sample, about 99% of these statements would be correct and 1% wrong.

In Text Question

The frequency distribution of the samples according to the values of the estimator based on the sample estimates. **True/False**

In Text Answer

l) Sampling and Non-sampling error

The error arising due to drawing inferences about the population on the basis of observations on a part (sample) of it is termed sampling error. The sampling error is non-existent in a complete enumeration survey since the whole population is surveyed.

The errors other than sampling errors such as those arising through non-response, in-completeness and inaccuracy of response are termed non-sampling errors and are likely to be more wide-spread and important in a complete enumeration survey than in a sample survey. Non-sampling errors arise due to various causes right from the beginning stage when the survey is planned and designed to the final stage when the data are processed and analyzed.

The sampling error usually decreases with increase in sample size (number of units selected in the sample) while the non-sampling error is likely to increase with increase in sample size.

As regards the non-sampling error, it is likely to be more in the case of a complete enumeration survey than in the case of a sample survey since it is possible to reduce the non-sampling error to a great extent by using better organization and suitably trained personnel at the field and tabulation stages in the latter than in the former.

Summary

In this study session you have learnt about:

(1) **Sampling** is a method of selecting a subset or part of a population that is representative of the entire population. Various sampling designs and techniques have been developed in an attempt to improve this representation.

(2) **Non-Probability Sampling Techniques:**

- **Quota Sampling:** This is one where, although the population is divided into identified groups, elements are selected from each group without recourse to randomness. Here the interviewer is free to use his discretion to select the units to be included in the sample.

- **Judgmental or Purposive Sampling:** This is a sample whose elementary units are chosen according to the discretion of expert who is familiar with the relevant characteristics of the population.

Sampling Concept:

1. Population
2. Sampling Unit
3. Sampling Frame
4. Random Sample
5. Non-random sample
6. Population parameters
7. Statistic, Estimator and Estimate
8. Sample design
9. Measures of error
10. Confidence interval
11. Sampling and Non-sampling error

Self-Assessment Questions (SAQs) for study session 2

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 2.1 (Testing Learning Outcomes 2.1)

Explain Sampling Techniques

SAQ 2.2 (Testing Learning Outcomes 2.2)

Discuss non Probability Sampling

SAQ 2.3 (Testing Learning Outcomes 2.3)

Discuss the following:

- Population
- Unit Sampling

References

Adamu, S. O. and Johnson, T. L.(1997) *Statistics for Beginners.* Ibadan: Book I. SAAL Publications

John, E. F.(1974). *Modern Elementary Statistics*, London: International Edition. Prentice Hall.

Murray, R. S. (1972) *Schaum's Outline Series. Theory and Problems of Statistics*. New York: McGraw-Hill Book Company

Olubusoye O. E. et al (2002) *Statistics for Engineering, Physical and Biological Sciences*. Ibadan: A Divine Touch Publication

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). *Statistical Theory and Methods* Ibadan: Joytal. Press.

Study Session 3: Large Sample Distribution of Means and Difference of Means

Introduction

The aim of this study session is to introduce you to what large sample distribution of means and difference of means are all about. Also we shall work examples on large sample distribution of means and difference of means.

Learning Outcomes for Study Session 3

At the end of this study session, you should be able to:
3.1 Discuss the Central Limit Theorem

3.2 Explain the Variance of the Sampling Distribution of Means: Parameter Known

3.3 Highlight on Large Sample Distribution of Means

3.1 The Central Limit Theorem

The Central Limit Theorem provides us with a shortcut to the information required for constructing a sampling distribution.

By applying the Theorem we can obtain the descriptive values for a sampling distribution (usually, the mean and the standard error, which is computed from the sampling variance) and we can also obtain probabilities associated with any of the sample means in the sampling distribution.

In other words, if n is large, the sampling distribution of the statistic $Z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}}$ can be approximated closely with the standard normal distribution. It is difficult to state precisely how large n must be so that this theorem applies; unless the distribution of the population has a very unusual shape. However, the approximation will be good even if n is relatively small- certainly, if n is 30 or more.

Instead we will focus on its major principles.

They are summarized below:

If we randomly select all samples of some size N out of a population with some mean M and some variance of ó 2 , then

- The mean of the sample means ( ) will equal M, the population mean;
- The sampling variance will be here (the population variance divided by N , the sample size). The standard error will equal the square root of the sampling variance;
- The sampling distribution of sample means will more closely approximate the Normal Distribution as N increases.

We'll discuss each of these points separately in the following sections of this chapter. But before we do, let us be sure to emphasize the three assumptions of the Central Limit Theorem: we know what size sample (N) we would like to draw from the population, and, more importantly, that we know the two population parameters M and ó 2.

3.1.1 The Mean of the Sampling Distribution of Means: Parameter Known

According to the Central Limit Theorem, the mean of the sampling distribution of means is equal to the population mean. We have already observed this in the examples given in the previous chapter.

Our population, consisting of the values 5, 6, 7, 8 and 9, has a mean of 7. When we took all samples of N = 2 or N = 3 out of this population, the mean of all the resulting sample means ( ) in the two sampling distributions were both equal to 7.

Therefore, if we know the parameter mean, we can set the mean of the sampling distribution equal to M .This allows us to avoid two massively difficult steps: (1) calculating sample means for all possible samples that can be drawn from the population and (2) calculating the sampling distribution mean from this mass of sample means.

In Text Question

By applying the Theorem we can obtain the descriptive values for a sampling **frequency**. **True\False**

In Text Answer

False (distribution)

3.2 The Variance of the Sampling Distribution of Means: Parameter Known

According to the Theorem, the variance of the sampling distribution of means equals the population variance divided by N, the sample size. The population variance (ó) and the size of the samples (N) drawn from that population have been identified in the preceding chapter as the two key factors which influence the variability of the sample means.

As we saw in the examples in that chapter, the larger the variance of the values in the population, the greater the range of values that the sample means can take on. We also saw that the sample size was inversely related to the variability of Sampling, Measurement, Distributions, and Descriptive Statistics sample means: the greater the sample size, the narrower the range of sample means.

The effect of both factors is thus captured by computing the value of the sampling variance as ó2/ N. If we know the variance of the population as well as the sample size, we can determine the sampling variance and the standard error.

This aspect of the theorem can be illustrated by using our running example. As you can see in Table3.1, the variance of the population equals 2.00. Applying the Central Limit Theorem to sample sizes of N= 2 and N = 3 yields the sampling variances and standard errors shown in Table 3.1. For N = 2 and N= 3,

Table 3.1 also shows the sampling variance and standard error for a sampling distribution based on a sample size of N = 4 drawn from the same population. Had we calculated these values from the set of 625 sample means, we would have obtained exactly the same results for the variance and standard error.

But what do we do when the population parameters are unknown? For example, assume that We are interested in studying the population of newly married couples. Specifically, we are interested in the amount of time they spend talking to each other each week about their relationship. It is highly unlikely that any parameters for this population would be available.

As we have already mentioned several times, the absence of known parameters is very common in communication research. How are we to proceed under these conditions? In the absence of known parameters we will have to make do with reliable estimates of these parameters. Such reliable estimates can be obtained when we take random samples of sufficient size from the population.

Suppose that we draw a random sample of N = 400 from this population. After measuring the amount of time these newlyweds spend talking to one another about their relationship we observe the mean to be 2 hours per week and the sample standard deviation is 1 hour per week. You will use this information to estimate the mean, the variance and the standard error or the sampling distribution.

In Text Question

Descriptive Statistics sample means: the greater the sample size, the narrower the range of sample means. **True/False**

In Text Answer

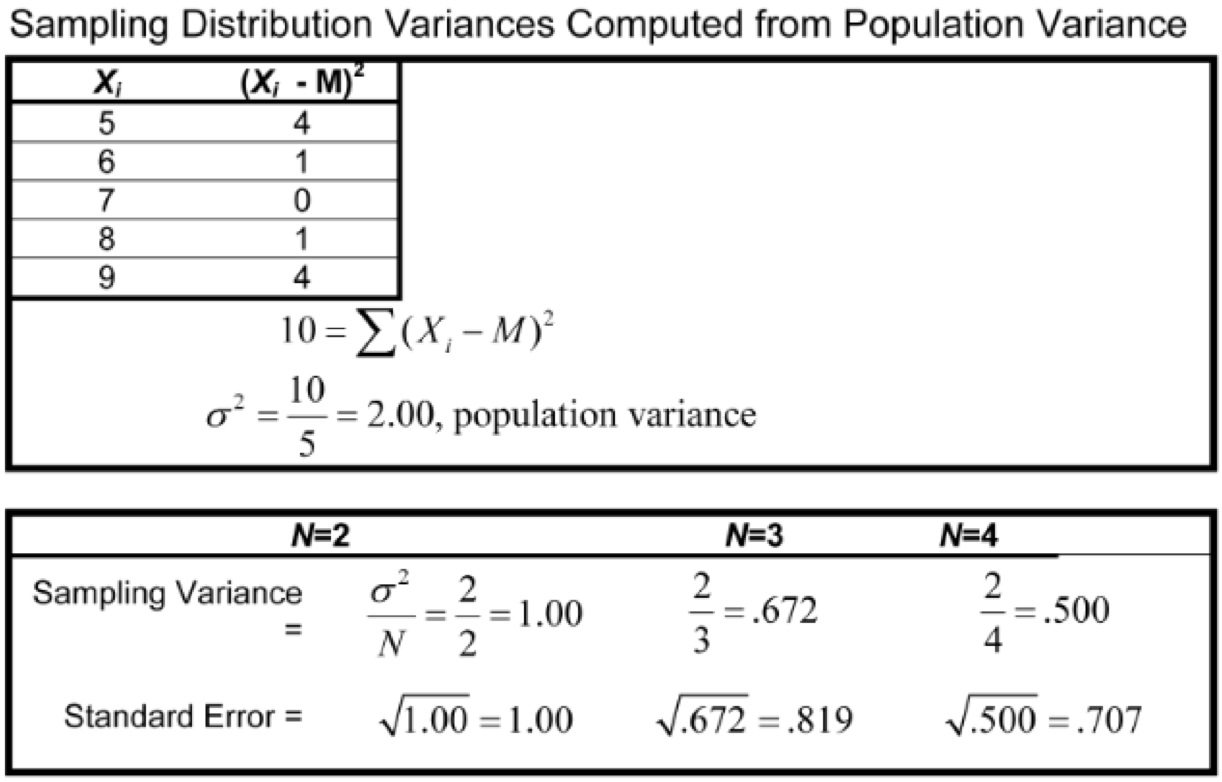
True

3.2.1 The Mean of the Sampling Distribution of Means: Parameter Unknown

Since we have only a single sample mean, we can’t compute the mean of the means. But we can make a simple assumption, based on probability that will allow us to work from the results of this single sample.

You know that the most probable mean found in the sampling distribution is the true population mean and that this mean is at the center

Table 3.1



Sampling Distribution Variances Computed from Population Variance

X_i	$(X_i - M)^2$
5	4
6	1
7	0
8	1
9	4

$$10 = \sum (X_i - M)^2$$

$$\sigma^2 = \frac{10}{5} = 2.00, \text{ population variance}$$

	N=2	N=3	N=4
Sampling Variance =	$\frac{\sigma^2}{N} = \frac{2}{2} = 1.00$	$\frac{2}{3} = .672$	$\frac{2}{4} = .500$
Standard Error =	$\sqrt{1.00} = 1.00$	$\sqrt{.672} = .819$	$\sqrt{.500} = .707$

of the sampling distribution. So if we have only one sample from a population, the assumption that the value of the sample mean is the same as the value of the population mean is more likely to be correct than any other assumption we could make. When we do this, we place the center of the sampling distribution right at the sample mean.

That is, we arrange our sampling distribution around the computed value of our sample mean. It is important to note that the sample mean of 2.0 is the best estimate of the unknown population (or true) mean. But we have to realize that there is also the possibility that the true population mean is somewhat higher or lower than that figure.

You can use the sampling distribution to describe how probable it is that the real population means falls somewhere other than the computed sample mean.

3.2.3 The Variance of the Sampling Distribution of Means: Parameter Unknown

The sampling variance (and hence the standard error) can be estimated from the sample variance if we are willing to make the following assumption.

If we are willing to assume about the population variance what we assumed about the population mean, namely, that the most probable value for this unknown parameters is the one which we have computed from our sample, we can again work with the results of our single sample.

Sampling Variance= var /

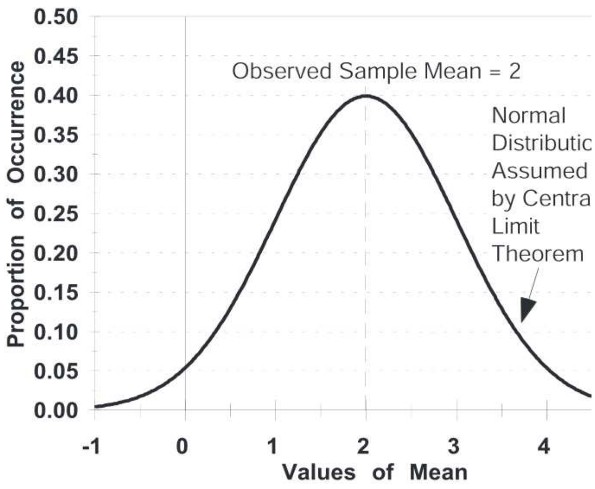
N

= 1 / 400 = .0025

$$\text{Std Error} = \sqrt{\text{var}/N} = \sqrt{1/400} = 0.05hrs$$

$$\text{Std Error} = sd/\sqrt{N} = 1/\sqrt{400} = 1/20 = 0.05hrs$$

Table 3.2



You now have a complete description of the sampling distribution, constructed from the information provided by a single random sample. Once the sampling distribution has been identified, either by using known parameters, or by using estimates of these parameters obtained from samples, we can now use this distribution to carry out the next important step: computing the probabilities of the means in the sampling distribution.

You need these probabilities to be able to make statements about the likelihood of the truth or falsity of our hypotheses, as we've already mentioned in the previous chapter. Whether the sampling distribution was derived from known parameters or from estimates matters little in terms of the remainder of the discussion in this chapter, as long as one condition is met: if estimates are used, the sample from which the estimates are derived should be sufficiently large.

A violation of this condition does not alter the logic, but requires some computational adjustments. Any statistics text will show you how to carry out these adjustments whenever small samples are encountered.

3.3 Large Sample Distribution of Means

Suppose a random sample of size n$_A$ from population A yield a mean $\overline{X}_A$; we know that provided n$_A$ is large and sampling is done from a finite population and it is done with replacement then $\overline{X}_A \sim N(\mu_A, \sigma_A^2 / n_A)$ and

$$Z = \frac{\overline{X}_A - \mu_A}{\sqrt{\frac{\sigma_A^2}{n_A}}}$$

which has the standard normal distribution; μ and σ being population mean and standard deviation respectively. When the population variance σ^2 is not known but n is sufficiently large, we may use $\hat{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ in its place. The approximation is still good.

If sampling is done without replacement, the Z value becomes

$$Z = \frac{\overline{X}_A - \mu_A}{\sqrt{\frac{\sigma_A^2}{n_A}(\frac{N - n_A}{N-1})}}$$

Example 1

In a savings bank, the average deposit is N160.5 with a standard deviation of N17.5. What is the probability that a group of 200 accounts taken will show an average deposit

1. of more than N165?
2. between 158.00 and 164.00

Solution

1. $P(\overline{X} > 165) = 1 - P(\overline{X} \leq 165)$

$$=1-P(Z\le\frac{165-160.5}{\sqrt{\frac{17.5^2}{200}}})$$

$=1-\varphi(3.6386)$

=1-0.9999

=0.0001

2. $P(158\le\bar{X}\le164)$

$$=P(\frac{158-160.5}{\sqrt{\frac{17.5^2}{200}}}\le Z\le\frac{164-160.5}{\sqrt{\frac{17.5^2}{200}}})$$

$$=\varphi(\frac{164-160.5}{1.24})-\varphi(\frac{158-160.5}{1.24})$$

$=\varphi(2.830)-\varphi(-2.018)$

$=\phi(2.830)-1+\phi(2.018)$

= 0.9977-1+0.978

Note

0.9977 is the value for 2.830 from the Z- normal table likewise 0.978 $\phi(-Z)=1-\phi(Z)$

3.3.1 Large Sample distribution of difference of means

If a random sample of size n_x taken from an X-population yields mean $\bar{X}$ and an independent one of size n_Y from a Y-population yields mean $\bar{Y}$,we know that the sampling distribution of the difference $\bar{X}-\bar{Y}$ is normal with mean $\mu_{\bar{X}-\bar{Y}}=\mu_X-\mu_Y$ the difference of the population means and variance $\sigma^2_{\bar{X}-\bar{Y}}=\sigma^2_{\bar{X}}+\sigma^2_{\bar{Y}}=\sigma^2_X/n_X+\sigma^2_Y/n_Y$ where $\sigma^2_{\bar{X}}$ is the variance of X- population and $\sigma^2_{\bar{Y}}$ is the variance of the Y-population. i.e. $\bar{X}-\bar{Y}\sim N\left(\mu_X-\mu_Y,\sigma^2_X/n_X+\sigma^2_Y/n_Y\right)$.

$$Z=\frac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{\sqrt{\left(\sigma^2_X/n_X+\sigma^2_Y/n_Y\right)}}\sim N(0,1)$$

When σ^2_X and σ^2_Y are not known, provided n_x and n_Y are sufficiently large, they may be estimated by

$$\hat{s}^2_X=\frac{1}{n_X}\sum_{i=1}^{n_X}(X-\bar{X})^2\ ,\ \hat{s}^2_Y=\frac{1}{n_Y}\sum_{i=1}^{n_Y}(X-\bar{X})^2$$

Example 2

The mean weight of an army is 70kg with a standard deviation of 5kg. What is the probability that 2 independent random groups of soldiers' consisting of 36 and 45 from the army will differ in their mean weight by

1. 2kg or less
2. 1kg or more

Solution

Let $\overline{X}_1, \overline{X}_2$ respectively denote the sample mean

(1) $P\left[\left|\overline{X}_1 - \overline{X}_2\right| \leq 2\right]$

$=\mathrm{P}\left[-2 \leq \overline{X}_1 - \overline{X}_2 \leq 2\right]$

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})}}$$

$$= P\left[\frac{-2}{\sqrt{25(\frac{1}{36} + \frac{1}{45}}} \leq \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{25(\frac{1}{36} + \frac{1}{45}}} \leq \frac{2}{\sqrt{25(\frac{1}{36} + \frac{1}{45})}}\right]$$

$$= \varphi\left[\frac{2}{\sqrt{25(\frac{1}{36} + \frac{1}{45}}}\right] - \varphi\left[\frac{-2}{\sqrt{25(\frac{1}{36} + \frac{1}{45})}}\right]$$

$$= 2\varphi(\frac{2}{\sqrt{25(\frac{1}{36} + \frac{1}{45})}}) - 1$$

$=2\,\varphi(1.78885) - 1 = 0.9264$

(2) $P\left[\left|\overline{X}_1 - \overline{X}_2\right| \geq 1\right] = 1 - P\left[\left|\overline{X}_1 - \overline{X}_2\right| < 1\right]$

$$=1 - \left[\frac{-1}{\sqrt{25(\frac{1}{36} + \frac{1}{45}}} < Z < \frac{1}{\sqrt{25(\frac{1}{36} + \frac{1}{45})}}\right]$$

$=1 - \left[2\varphi(0.8944) - 1\right] = 0.3711$

Summary

In this study session you have learnt about:

1. **The Central Limit Theorem**

   The Central Limit Theorem provides us with a shortcut to the information required for constructing a sampling distribution.

2. **The Variance of the Sampling Distribution of Means: Parameter Known**

   According to the Theorem, the variance of the sampling distribution of means equals the population variance divided by N, the sample size.

   The population variance (ó) and the size of the samples (N) drawn from that population have been identified in the preceding chapter as the two key factors which influence the variability of the sample means.

3. **Large Sample Distribution of Means**

Suppose a random sample of size n_A from population A yield a mean $\overline{X}_A$; we know that provided n_A is large and sampling is done from a finite population and it is done with replacement

Self-Assessment Questions (SAQs) for study session 3

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 3.1 (Testing Learning Outcomes 3.1)

Briefly Discuss on Central Limit Theorem

SAQ 3.2 (Testing Learning Outcomes 3.2)

Enumerate the Variance of the Sampling Distribution of Means: Parameter Known

SAQ 3.3 (Testing Learning Outcomes 3.3)

Explain Large Sample Distribution of Means

References

John, E. F.(1974). *Modern Elementary Statistics*, London: International Editions Prentice Hall

Murray, R. S. (1972) *Schaum's Outline Series. Theory and Problems of Statistics*. New York: McGraw-Hill Book Company

Olubusoye, O. E. et al (2002) *Statistics for Engiineering, Pyhsical and Biological Sciences*. Ibadan: A Divine Touch Publication.

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Study Session 4: Large Sample Distribution of Proportion and Difference of Proportions

Introduction

The mean of the distribution of sample proportions is equal to the population proportion (p). The standard deviation of the distribution of sample proportions is symbolized by S E (p ^) and equals p (1 − p) n ; this is known as the standard error of p ^ .

The aim of this study session is to introduce you to what large sample distribution of proportion and difference of proportions are all about. You shall also work examples on large sample distribution of proportion and its difference.

Learning Outcomes for Study Session 4

4.1 Explain large Sample Distribution of Proportion

4.2 Large Sample Distribution of Difference of Proportions

4.1 Large Sample Distribution of Proportion

What is the mean of the sampling distribution of the sample proportion?

The **mean** of the **distribution** of **sample proportions** is equal to the population **proportion**, p . If p is unknown, we estimate it using p ^ . The standard deviation of the **distribution** of **sample proportions** is symbolized by S E (p ^) and equals p (1 − p) n ; this is the standard error of p ^ .

Let p be the sample proportion obtained from a random sample of size n and p the population proportion. Assume that the population is infinite or if it is finite assume that sampling is done with replacement.

$$Z = \frac{P - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$$

which has the standard normal distribution

Note: $p_0 + q_0 = 1$. If sampling is done without replacement Z becomes

$$Z = \frac{P - p_0}{\sqrt{\frac{p_0 q_0}{n}(\frac{N-n}{N-1})}}$$

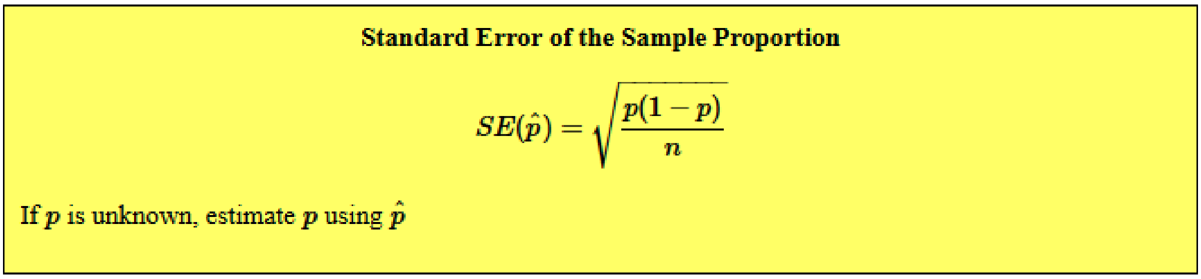
4.1.1 Rule of Sample Proportions (Normal Approximation Method)

If samples of the same size (*n*) are repeatedly randomly drawn from a population, and the proportion of successes in each sample is recorded (*p*ˆ), the distribution of the sample proportions (i.e., the sampling distribution) can be approximated by a normal distribution given that both *n*×*p*≥10 and *n*×(1−*p*)≥10.

This is known as the **Rule of Sample Proportions.** Note that some textbooks use a minimum of 15 instead of 10.

The mean of the distribution of sample proportions is equal to the population proportion (*p*). The standard deviation of the distribution of sample proportions is symbolized by *SE*(*p*ˆ) and equals *p*(1−*p*)*n*−−−−−√; this is known as the **standard error of *p*ˆ**. The symbol *σp*ˆ is also used to signify the standard deviation of the distribution of sample proportions.

Table 4.1: Standard Error

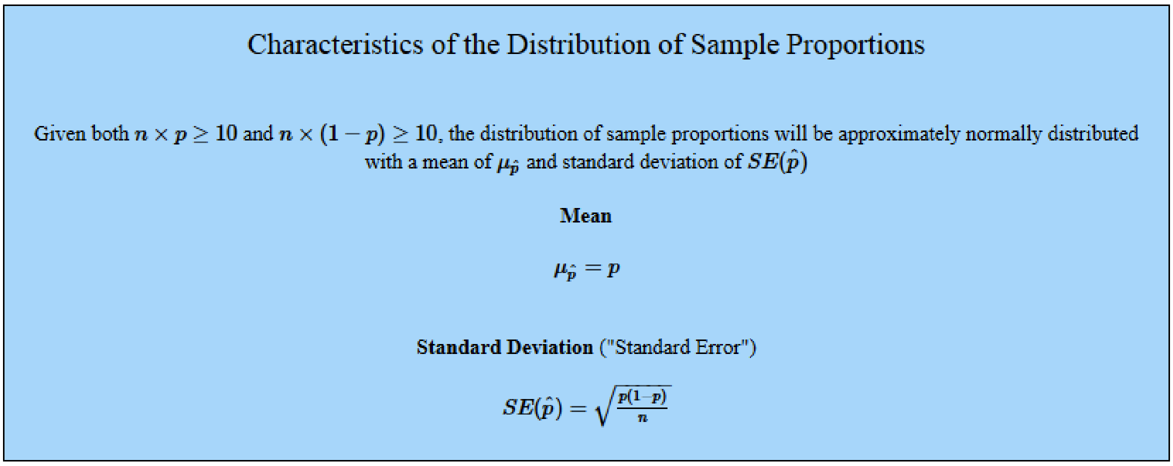


Standard Error of the Sample Proportion

$$SE(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

If p is unknown, estimate p using $\hat{p}$

The box below summarizes the rule of sample proportions:



Characteristics of the Distribution of Sample Proportions

Given both $n \times p \geq 10$ and $n \times (1-p) \geq 10$, the distribution of sample proportions will be approximately normally distributed with a mean of $\mu_{\hat{p}}$ and standard deviation of $SE(\hat{p})$

Mean

$$\mu_{\hat{p}} = p$$

Standard Deviation ("Standard Error")

$$SE(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

Example 1

If David classifies his yams by weight as large and small and finds that 40% are large, find the probability that in a random sample of 150 yams from David’s farm

i. more than 33% are large

ii. more than 30% but less than 40% are large

Solution

i. $P[P > 0.33] = 1 - P[P \leq 0.33]$

$$=1-P\left[\frac{P-p}{\sqrt{\frac{pq}{n}}} \leq \frac{P-0.4}{\sqrt{\frac{0.4\times0.6}{150}}}\right]$$

$$=1-p\left[\frac{0.33-0.4}{\sqrt{\frac{0.4\times0.6}{150}}}\right] \qquad =1-\varphi(\frac{-0.067}{0.04})$$

$$=1-[1-\varphi(1.675)]$$

= 0.9530

ii. $P(0.30 < p < 0.42)$

$$= P\left[\frac{0.3-0.4}{0.04} < Z < \frac{0.42-0.4}{0.04}\right]$$

$$= \varphi(\frac{0.42-0.4}{0.04}) - \varphi(\frac{0.3-0.4}{0.04})$$

$$= \phi(0.5) - 1 + \phi(2.5)$$

= 0.6853

4.2 Large Sample Distribution of Difference of Proportions

Statistics problems often involve comparisons between two independent sample proportions. This lesson explains how to compute probabilities associated with differences between proportions.

4.2.1 Difference Between Proportions: Theory

Suppose we have two populations with proportions equal to P_1 and P_2. Suppose further that we take all possible samples of size n_1 and n_2. And finally, suppose that the following assumptions are valid.

- The size of each population is large relative to the sample drawn from the population. That is, N_1 is large relative to n_1, and N_2 is large relative to n_2. (In this context, populations are considered to be large if they are at least 20 times bigger than their sample.)
- The samples from each population are big enough to justify using a normal distribution to model differences between proportions. The sample sizes will be big enough when the following conditions are met: $n_1P_1 \geq 10$, $n_1(1 - P_1) \geq 10$, $n_2P_2 \geq 10$, and $n_2(1 - P_2) \geq 10$.

  (This criterion requires that at least 40 observations be sampled from each population. When P_1 or P_1 is more extreme than 0.5, even more observations are required.)

- The samples are independent; that is, observations in population 1 are not affected by observations in population 2, and vice versa.

Given these assumptions, we know the following:

- The set of differences between sample proportions will be normally distributed. We know this from the central limit theorem.
- The expected value of the difference between all possible sample proportions is equal to the difference between population proportions. Thus, E(p_1 - p_2) = P_1 - P_2.
- The standard deviation of the difference between sample proportions (σ_d) is approximately equal to:

Example 2
A bag contains 60 ripe and 40 unripe bananas after two independent random samples; each of 50 bananas is drawn from the bag. Find the probability that the number of ripe banana in the sample differs by more than 6.

Solution

$$P\left[|P_A - P_B| > \frac{6}{50}\right] = 1 - P\left[|P_A - P_B| \le \frac{6}{50}\right]$$

Using the Z statistic, we have

$$= 1 - \left[\phi\left(\frac{\frac{6}{50}}{\sqrt{\frac{2\times 0.6\times 0.4}{50}}}\right) - \phi\left(\frac{\frac{-6}{50}}{\sqrt{\frac{2\times 0.6\times 0.4}{50}}}\right)\right]$$

$$= 2\left[1 - \varphi(1.328)\right] = 0.1845$$

Summary

In this study session you have learnt about:

1. **Large Sample Distribution of Proportion**

   What is the mean of the sampling distribution of the sample proportion?

   The mean of the distribution of sample proportions is equal to the population proportion, p . If p is unknown, we estimate it using p ^ . The standard deviation of the distribution of sample proportions is symbolized by S E (p ^) and equals p (1 − p) n ; this is the standard error of p ^

2. Large Sample Distribution of Difference of Proportions

Statistics problems often involve comparisons between two independent sample proportions. This lesson explains how to compute probabilities associated with differences between proportions.

Self-Assessment Questions (SAQs) for study session 4

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 4.1 (Testing Learning Outcomes 4.1)

Discuss on Large Sample Distribution of Proportion

SAQ 4.2 (Testing Learning Outcomes 4.2)

Explain the Large Sample Distribution of Difference of Proportions

References

John, E. F.(1974). Modern Elementary Statistics, International. London: Editions Prentice Hall

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications

Shangodoyin, D. K. et al (2002). Statistical Theory and Methods Ibadan: Joytal Press.

Study Session 5: Introduction to Estimation

Introduction

In actual applications, the values of the parameters will not be known. You turn your attention now from probability to statistics. Statistics deals with the problem of estimating parameters and making inference about the values of parameters based on data

The focus of this study session is on the theory of estimation. You will be introduces to nature of estimates that are bound to be encountered in everyday usage. You will also be exposed to confidence interval estimate of population parameters and confidence coefficient.

Learning Outcomes for Study Session 5

At the end of this study session, you should be able to:
5.1 Define Estimation

5.2 Explain the Interval Estimate

5.1 Definition of Estimation

Estimation is a process by which a statistic (summary of a collection of numerical data e.g. total, range, average, etc.) obtained from a sample is used to estimate the parameters of the population from which the sample has been drawn. Its need arises in practically every statistical decision-making in all spheres of life. The following are the nature of estimates we are bound to encounter in everyday usage.

5.1.1 Unbiased Estimate

A statistic is called an unbiased estimator of a population parameter if the mean or expectation of the statistic is equal to the parameter: $E(\overline{X}) = \mu$

The corresponding value of the statistic is then called an unbiased estimate of the parameter.

5.1.2 Efficient Estimate

If the sampling distributions of two statistics have the same mean, the statistic with smaller variance is called a more efficient estimator of the mean. The corresponding value of the efficient statistic is then called an efficient estimate. Clearly in practice, the target is to have estimates, which are both efficient and unbiased, though it is not always possible.

5.1.3 Point Estimate

This is an estimate of a population parameter, which is given by a single numerical value e.g. mean, $\overline{x} = \frac{\sum xi}{n}$ is a point estimate for μ.

Point Estimates of Population Parameters

From the sample, a value is calculated which serves as a point estimate for the population parameter of interest.

a) The best estimate of the population **percentage**, π, is the sample percentage, p.

b) The best estimate of the unknown population **mean**, μ, is the sample mean, $\overline{\mathrm{x}} = \frac{\sum \mathrm{x}}{\mathrm{n}}$.

This estimate of μ is often written $\hat{\mu}$ and referred to as 'mu hat'.

c) The best estimate of the unknown population **standard deviation**, σ, is the sample standard deviation s, where:

$$\mathrm{s} \quad = \quad \sqrt{\frac{\sum(\mathrm{x}-\overline{\mathrm{x}})^2}{(\mathrm{n}-1)}}$$

This is obtained from the $X_{\sigma_{n-1}}$ key on the calculator.

N.B. $\mathrm{s} \quad = \quad \sqrt{\frac{\sum(\mathrm{x}-\overline{\mathrm{x}})^2}{(\mathrm{n})}}$

from X_{σ_n} is **not** used as it underestimates the value of σ.

5.2 Interval Estimate

An estimate of a population parameter given by two numerical values between which the parameter may be considered to lie with a given probability is called an interval estimate of the parameter, that is $(\overline{x} - error \le \mu \le \overline{x} + error)$ is an interval estimate. For instance, with 95%, the population μ will be between $\overline{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

5.2.1 Interval Estimate of Population Parameter (Confidence interval)

Sometimes it is more useful to quote two limits between which the parameter is expected to lie, together with the probability of it lying in that range.

The limits are called the **confidence limits** and the interval between them the **confidence interval.**

The width of the confidence interval depends on three sensible factors:

a) The degree of confidence we wish to have in it,
   i.e. the probability of it including the 'truth', e.g. 95%;

b) The size of the sample, n;

c) The amount of variation among the members of the sample, e.g. for means this the standard deviation.

The **confidence interval** is therefore an interval centers on the point estimate, in this case either a percentage or a mean, within which we expect the population parameter to lie.

The width of the interval is dependent on the confidence we need to have that it does in fact include the population parameter, the size of the sample, n, and its standard deviation, s, if estimating means. These last two parameters are used to calculate the **standard error**, s/√n, which is also referred to as the standard deviation of the mean.

The number of standard errors included in the interval is found from statistical tables - either the normal or the t-table. Always use the normal tables for percentages which need large samples. For means the choice of table depends on the sample size and the population standard deviation:

Table 5.1: Population Standard Deviation

Sample size	Population standard deviation Known: standard error = $\frac{\sigma}{\sqrt{n}}$	 Unknown: standard error = $\frac{s}{\sqrt{n}}$
Large	Normal tables	Normal tables
Small	Normal tables	t-tables

5.2.2 Interpretation of Confidence intervals

How do we interpret a confidence interval? If 100 similar samples were taken and analyzed then, for a 95% confidence interval, we are confident that 95 of the intervals calculated would include the true population mean. In practice we tend to say that we are 95% confident that our interval includes the true population value. Note that there is only one true value for the population mean, it is the variation between samples which gives the range of confidence intervals.

In Text Question

The confidence interval is therefore an interval centers on the vertical estimate. **True/False**

In Text Answer

False (Point)

5.2.3 Confidence Intervals for a Percentage or Proportion

The only difference between calculating the interval for percentages or for proportions is that the former total 100 and the latter total 1. This difference is reflected in the formulae used, otherwise the methods are identical. Percentages are probably the more commonly calculated so in Example 2 we will estimate a population percentage.

The confidence interval for a population percentage or a proportion, π, is given by:

$$\pi = p \pm z\sqrt{\frac{p(100-p)}{n}} \text{ for a percentage or } \pi = p \pm z\sqrt{\frac{p(1-p)}{n}} \text{ for a proportion}$$

where: π is the unknown population percentage or proportion being estimated,

p is the sample percentage or proportion, i.e. the point estimate for π,

z is the appropriate value from the normal tables,

n is the sample size.

The formulae $\sqrt{\frac{p(100-p)}{n}}$ and $\sqrt{\frac{p(1-p)}{n}}$ represent the standard errors of a percentage and a proportion respectively.

The samples must be large, (>30), so that the normal table may be used in the formula.

We therefore estimate the confidence limits as being at z standard errors either side of the sample percentage or proportion.

The value of z, from the normal table, depends upon the degree of confidence, e.g. 95%, required. We are prepared to be incorrect in our estimate 5% of the time and confidence intervals are always symmetrical so, in the tables we look for Q to be 5%, two tails.

Example 1

If we say a distance is 5.15 metres, we are giving a point estimate. If, on the other hand, we say that the distance is 5.15 ± 0.12 metres, i.e. the distance between 5.03 and 5.27 metres, we are giving an interval estimate.

5.2.4 Interval Estimation

In this section, our aim is to estimate the interval to which the population parameter of interest lies. Given the sample mean, $\overline{X}$ and variance S^2, we intend to estimate the corresponding population parameters, that is, μ and σ^2.

Assume that μs and σs are the mean and standard deviation (error) of the sampling distribution of a statistic S. Then, for the distribution to be normal, it is expected to lie in the interval μs ± σs, μs ± 2σs, μs ± 3σs. This gives about 68.27%, 95.45% and 99.73% of the time respectively.

Similarly, we can be confident of finding μs in the intervals S ± σs, S ± 2σs, S ± 3σs in about 68.27%, 95.45% and 99.73% of the time respectively. The respective intervals are called the 68.27%, 95.45% and 99.73% confidence intervals for estimating the population parameter, μs (in the case of an unbiased S).

The end numbers of the interval are called the confidence limits. The percentage confidence is called the confidence level. The numbers 1.0, 2.0, 3.0, 1.98, 2.58, etc in the confidence limits are called the critical values (or confidence coefficients). Because of repeated use, we are going to make use of certain areas under the standard normal curve, let us take a look at them.

$P(T-2\sigma_T \le \mu_T \le T+2\sigma_T) = 0.9545$ and $P(T-\sigma_T \le \mu_T \le T+\sigma_T) = 0.6827$

The statement $P(t_1 \le \mu_T \le t_2) = 1-\alpha$, say, means that we expect the interval $[t_1, t_2]$ to cover μ_T, $100(1-\alpha)\%$ of the time; which is to say that we are $100(1-\alpha)\%$ confident that $[t_1, t_2]$ covers μ_T. $[t_1, t_2]$ is called a confidence interval for estimating μ_T. The end points of the interval are called confidence limits. The amount of confidence in any interval is called the confidence coefficient.

Summary

In this study session you have learnt about:

1. Estimation

Estimation is a process by which a statistic (summary of a collection of numerical data e.g. total, range, average, etc.) obtained from a sample is used to estimate the parameters of the population from which the sample has been drawn.

2. Interval Estimate

An estimate of a population parameter given by two numerical values between which the parameter may be considered to lie with a given probability is called an interval estimate of the parameter

Self-Assessment Questions (SAQs) for study session 5

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 5.1 (Testing Learning Outcomes 5.1)

Define Estimation

SAQ 5.2 (Testing Learning Outcomes 5.2)

Explain Confidence interval

References

John, E. F. (1974). Modern Elementary Statistics, International. London: Edition. Prentice Hall.

Murray, R. S. (1972) Schaum’s Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Olubusoye O. E. et al (2002) Statistics for Engineering, Physical and Biological Sciences. Ibadan: A Divine Touch Publication

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). *Statistical Theory and Methods* Ibadan: Joytal. Press.

Study Session 6: Large Sample Interval Estimation for Means and Proportions

Introduction

A point estimate of a population parameter is a single value of a statistic. For example, the sample mean x is a point estimate of the population mean µ. Similarly, the sample proportion p is a point estimate of the population proportion P. Interval estimate.

The aim of this study session is to introduce you to large sample interval estimation of means and proportions. You shall also work examples on large sample estimation of means and proportions.

Learning Outcomes for Study Session 6

At the end of this study session, you should be able to:

6.1 Large Sample Estimation of a Population Mean

6.2 Large Sample Estimation of a Population Mean

6.1 Large Sample Interval Estimation for Mean

If the statistic S is the sample mean $\overline{x}$, then 95% and 99% confidence level for estimation of the population mean µ, are given by $\overline{x} \pm 1.96\sigma_{\overline{x}}$ and $\overline{x} \pm 2.58\sigma_{\overline{x}}$ respectively. Generally, the confidence limits are given as $\overline{x} \pm Z_i\sigma_x$, where Zi is the level of confidence desired and can be got from the table. The sample variance is then given as $\frac{\sigma^2}{n}$. Thus, the confidence interval for the population mean is then given as $\overline{x} \pm Z_c\frac{\sigma}{\sqrt{n}}$.

This becomes $(\overline{X} + Z_c\frac{\sigma}{\sqrt{n}}, \overline{X} - Z_c\frac{\sigma}{\sqrt{n}})$ and the confidence limit is without the brackets that is

$\bar{X} + Z_c \frac{\sigma}{\sqrt{n}}, \bar{X} - Z_c \frac{\sigma}{\sqrt{n}}$. This is where sampling is from an infinite population, or if it is with replacement from a finite population. But if sampling is without replacement from a population of size N, the confidence interval for the population mean is $(X \pm Z_c \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}})$

Note: If σ^2 is not known provided n the sample size is sufficiently large for an infinite population or when population is finite provided sampling is done with replacement we may use $S^2 = \frac{1}{n-1}\sum\left(X_i - \bar{X}\right)^2$ in place of σ^2, the approximation is still fair for the normal distribution.

Example 1

The mean weight of a random sample of 80 yams from a farm is 3.75kg with a standard deviation 0.85kg. Find (i) 95% (ii) (97.5%) (iii) 99% confidence interval for the population mean of all yams on the farm.

Solution: 95% = 0.95, α = 0.05, $Z_{1-\alpha/2} = Z_{0\cdot975}$ = 1.96

The confidence interval is = $\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$

The required C.I. for population mean is

i. $= \left(\bar{x} \pm 1.96\frac{S}{\sqrt{n-1}}\right)$

$= \left(3.75 \pm 1.96x\frac{0.85}{\sqrt{79}}\right)$

= (3.56, 3.94)

ii. We want 97.5% = 0.975 but α = 0.025, $\frac{\alpha}{2}$ = 0.0125

$Z_{1-\alpha/2} = Z_{0\cdot9875}$ = 2.24

The required confidence interval is $\left(3.75 - 2.24x\frac{0.85}{\sqrt{79}},\ 3.75 + 2.24x\frac{0.85}{\sqrt{79}}\right)$ = (3.54, 3.96)

iii. 99% = 0.99, $\frac{\alpha}{2} = \frac{0.01}{2}$, $Z_{1-\alpha/2} = Z_{0.995} = 2.58$ The required C.I. is

$$= \left(3.75 \pm 2.58x\frac{0.85}{\sqrt{79}}\right)$$

= (3.50, 4.00)

6.2 Large Sample Estimation of a Population Mean

The Central Limit Theorem says that, for large samples (samples of size $n \geq 30$), when viewed as a random variable the sample mean X−− is normally distributed with mean μX−−=μ and standard deviation σX−−=σ/n−−√. The Empirical Rule says that we must go about two standard deviations from the mean to capture 95% of the values of X−− generated by sample after sample.

A more precise distance based on the normality of X−− is 1.960 standard deviations, which is E=1.960σ/n−−√.

The key idea in the construction of the 95% confidence interval is this, as illustrated in Figure "When Winged Dots Capture the Population Mean": because in sample after sample 95% of the values of X−−lie in the interval [μ−E,μ+E], if we adjoin to each side of the point estimate x−− a “wing” of length E, 95% of the intervals formed by the winged dots contain μ.

The 95% confidence interval is thus x−−±1.960σ/n−−√. For a different level of confidence, say 90% or 99%, the number 1.960 will change, but the idea is the same.

When Winged Dots Capture the Population Mean

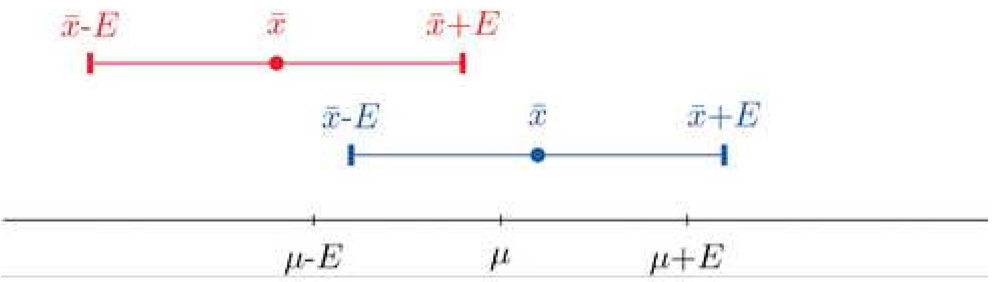


Figure 6.1: When Winged Dots Capture the Population Mean

"Computer Simulation of 40 95% Confidence Intervals for a Mean" shows the intervals generated by a computer simulation of drawing 40 samples from a normally distributed population and constructing the 95% confidence interval for each one. We expect that about $(0.05)(40)=2$ of the intervals so constructed would fail to contain the population mean μ, and in this simulation two of the intervals, shown in red, do.

It is standard practice to identify the level of confidence in terms of the area α in the two tails of the distribution of X−− when the middle part specified by the level of confidence is taken out.

This is shown in figure 6.2, drawn for the general situation, and in figure 6.2, drawn for 95% confidence. Remember from Section 5.4.1 "Tails of the Standard Normal Distribution" in study session 5 "Continuous Random Variables" that the z-value that a cut off a right tail of area c is denoted zc. Thus the number 1.960 in the example is z.025, which is zα⁄2 for α=1−0.95=0.05.

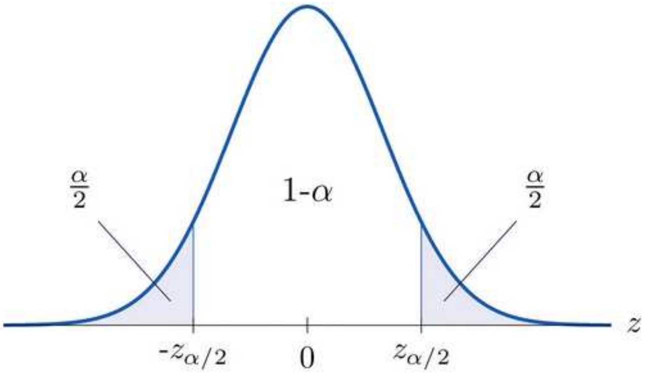
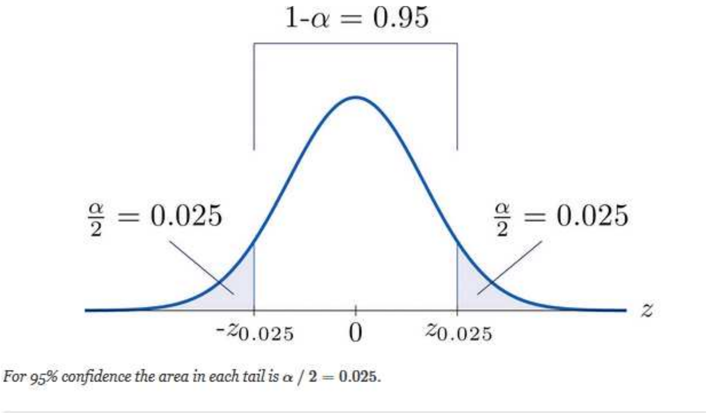


Figure 6.2: Normal Distribution For $100(1-\alpha)$% confidence the area in each tail is $\alpha/2$.



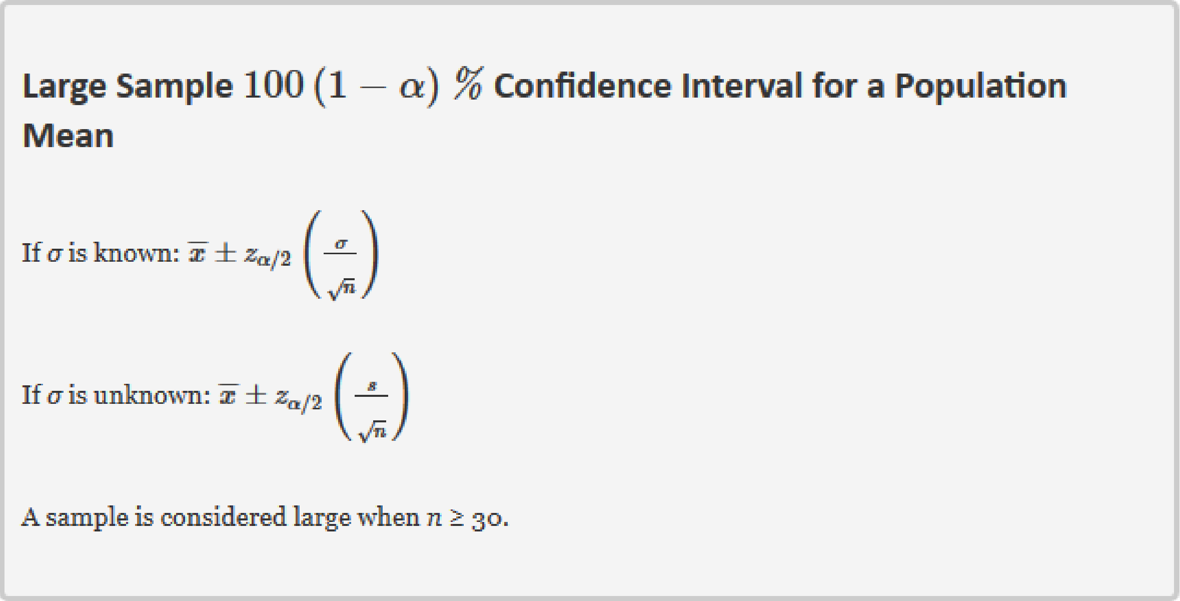
For 95% confidence the area in each tail is α / 2 = 0.025.

Figure 6.3: Normal Distribution For 95% confidence the area in each tail is $\alpha/2=0.025$.

The level of confidence can be any number between 0 and 100%, but the most common values are probably 90% ($\alpha=0.10$

), 95% ($\alpha=0.05$), and 99% ($\alpha=0.01$). Thus in general for a $100(1-\alpha)$ % confidence interval, $E=z_{\alpha/2}(\sigma/n{-}{-}\surd)$, so the formula for the confidence interval is $x{-}{-}\pm z_{\alpha/2}(\sigma/n{-}{-}\surd)$. While sometimes the population standard deviation σ is known, typically it is not. If not, for $n \geq 30$ it is generally safe to approximate σ by the sample standard deviation s.

Table 6.1: Table of Sample



Large Sample $100\,(1-\alpha)\,\%$ Confidence Interval for a Population Mean

If σ is known: $\bar{x} \pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$

If σ is unknown: $\bar{x} \pm z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$

A sample is considered large when $n \geq 30$.

As mentioned earlier, the number $E=z_{\alpha/2}\sigma/n{-}{-}\surd$ or $E=z_{\alpha/2}s/n{-}{-}\surd$ is called the *margin of error* of the estimate.

6.2.1 Large Sample Interval Estimation of Proportion

If T_n =p, the proportion exhibiting a certain attribute in a sample of size n and the proportion exhibiting the attribute in the population is p, the 100(1-α)% confidence interval for P is given by $P \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{pq}{n}}$, if sampling is from an infinite population or when it is from a finite population but sampling is done with replacement. If sampling is without replacement from a finite population of size N, the 100(1-α)% confidence interval is $P \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{pq}{n} \quad (\frac{N-n}{N-1})}$

where $Z_{1-\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$ th quantile of the standard normal distribution

Example 1

A random of 1000 eligible voters in a country reveals that 575 of them would want the National Anthem replaced. What is the 95% confidence limits for estimating the proportion of eligible voters in the country who would want a new Anthem.

Solution

P= 575/1000=0.575

95% confidence limits are

$$P \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{pq}{n}} = 0.575 \pm 1.96\sqrt{\frac{0.575 \times 0.425}{1000}}$$

that is 0.54436, 0.60564

Summary

In this study session you have learnt about:

1. **Large Sample Interval Estimation for Mean**

If the statistic S is the sample mean $\overline{x}$, then 95% and 99% confidence level for estimation of the population mean μ, are given by $\overline{x} \pm 1.96\sigma_{\overline{x}}$ and $\overline{x} \pm 2.58\sigma_{\overline{x}}$ respectively.

2. **Large Sample Estimation of a Population Mean**

The Central Limit Theorem says that, for large samples (samples of size $n \geq 30$), when viewed as a random variable the sample mean *X*−− is normally distributed with mean *μX*−−=*μ* and standard deviation *σX*−−=*σ*/*n*−−√.

Self-Assessment Questions (SAQs) for study session 6

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 6.1 (Testing Learning Outcomes 6.1)

Explain the Large Sample Interval Estimation for Mean

SAQ 6.2 (Testing Learning Outcomes 6.2)

Discuss the Large Sample Estimation of a Population Mean

References

John, E. F.(1974). Modern Elementary Statistics, London. International Edition. Prentice Hall.

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York:McGraw-Hill Book Company

Olubusoye O. E. et al (2002) *Statistics for Engineering, Physical and Biological Sciences*. Ibadan: A Divine Touch Publication

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). *Statistical Theory and Methods*. Ibadan: Joytal Press.

Study Session 7: Large Sample Interval Estimation for Difference of Means and Proportions

Introduction

Two-sample t-tests are among the most common statistical analyses performed to compare – as the name implies – two samples comprised of continuous data satisfying parametric assumptions. What are you actually doing, though, when you perform a t - test to compare sample means?

Usually, you want to know if the samples are from the same population (i.e. are the means statistically equal, based on your confidence level?) or from different populations (i.e. are the means significantly different, based on your confidence level?). You perform a two-sample t-test to help you make that decision.

The aim of this study session is to introduce you to large sample interval estimation of difference of means and proportions. Furthermore, you shall work examples on large sample estimation of difference of means and proportions.

Learning Outcomes for Study Session 7

At the end of this study session, you should be able to:

7.1 Highlight on Two-Sample Test of Means

7.2 Explain When to Use Two-Sample t-Tests

7.3 Discuss on Two – Sample Test of Proportions – Large Sample

7.1 Two Sample Test of Means

When you perform a two-sample t- test, you are actually testing whether the difference between the sample means is equal to zero. How does that work conceptually? Well, if two samples were exactly the same, the difference in means would be equal to zero and the distributions would be perfectly overlapping.

For samples with identical means,$\mu1–\mu2= 0$ The distributions for identical samples would be perfectly overlapping. However, what happens as the samples begin to differ? Then, the difference in means is not equal to zero, and you start to see a separation of means. As that difference becomes greater, it becomes less and less likely that the two samples were taken from the same population.

At a large enough difference (and also depending on the sample spread, of course), you may decide that it is unlikely that the two samples are from the same population and you could conclude that the samples are significantly different (or that the difference in means in significantly different from zero).

For two-sample t-tests, the null and alternative hypotheses are as follows: H0 = Difference between sample means is equal to zero (i.e., sample n means do not differ significantly) H1 = Difference between sample means $\neq$ zero (i.e., sample means are significantly different)

Mathematically, how is the test statistic calculated (i.e., what is calculated to help you determine whether the difference in means is sufficiently greater than zero to make you think the samples are from different populations)?

Generally, the test statistic (t) is found as follows (with some variation based on equal sample variances, paired versus unpaired data, etc.): != Where $\mu1$ and $\mu2$ are the sample means, SD 1 and SD2 are the corresponding sample standard deviations, and n1 and n2 are the corresponding sample sizes. Calculating the test statistic (t) allows you to find the p-value, or probability, of finding a difference that large or a more extreme difference given that the null hypothesis is true.

In Text Question

When you perform a two-sample t- test, you are actually testing whether the difference between the sample means is equal to

(a) Zero

(b) One

(c) Two

(d) Three.

In Text Answer

The answer is (a) Zero

7.2 When to Use Two-Sample t-Tests

You can use two-sample t-tests to determine whether two samples are likely or unlikely to have been selected from the same population (i.e., to decide whether two samples differ significantly based on the selected confidence level).Two- sample t-tests are, at times, overused in place of other more appropriate statistical tests because they are very familiar to many researchers.

However, be careful – there are specific types of data for which t-tests are appropriate, and many for which they aren't.

Two-sample t-tests are appropriate when:

- You are only comparing TWO samples (if > 2, consider ANOVA or other multi-sample test to avoid increased likelihood of Type I error occurring)
- Samples are from normally distributed populations (see document re: tests for normality)
- Samples are randomly selected and independent
- Sample data are at least interval or ratio level data (differences between values are meaningful)

To perform two-sample t- tests, your samples do not need to:

- Have equal variances (Welch approximation)
- Have equal sample size
- Have a minimum sample size, so long as the assumptions hold (this is one of the greatest things about a t-test) Paired versus Unpaired Data

When performing two-sample t-tests, it is important to know whether your data is paired or unpaired.

In paired data, each data point in one sample is associated with (or related to) a single data point in the second sample. The data sets cannot be analyzed as completely independent data sets, because of the point-to-point associations between the sample data.

Examples of experiments with paired data are:

- Sample 1 taken at Time 0 days for 20 people (before treatment) to measure cholesterol.
- Sample 2 taken at Time 30 days for the same 20 people (following treatment).
- A experiment with 30 sets of twins (one male, one female) to investigate sex-dependent brain development.

Alternatively, unpaired data are data from samples where there is no relation or association between a data point in one sample set and any data point in another sample set.

Examples of experiments with unpaired data are:

- A comparison of blood sugar levels in diabetic patients versus non-diabetic patients
- An experiment in which 15 petri -plates of bacteria are treated with chlorine, and 15 are treated with EDTA
- Measuring zinc oxide concentrations in mussels from a shoreline in California and from a shoreline in Oregon.

In Text Question

Two-sample t-tests are appropriate when Samples are randomly selected and independent. **True/False**

In Text Answer

True

7.3 Two – Sample Test of Proportions – Large Sample

Confidence Intervals to Estimate the Difference between Two Population Proportions: p1−p2

Point estimate is the difference between the two sample proportions, written as:

p^1−p^2=y1n1−y2n2

The mean of its sampling distribution is p1−p2 and the standard deviation is given by:

p1^(1−p1^)n1+p2^(1−p2^)n2−−−−−−−−−−−−−−−−−−−√

When the observed number of successes and the observed number of failures are greater than or equal to 5 for both populations, then the sampling distribution of p^1−p^2 is approximately normal and we can use z-methods.

In the following, the formula of the confidence interval and the test statistic are given for reference. You can use Minitab to perform the inference. It is more important to recognize the problem and be able to use Minitab to draw a conclusion than to train yourself to use the tedious formula. The 100(1−α) % confidence interval of p1−p2 is given by:

p^1−p^2±zα/2·s.e.(p^1−p^2) where s.e.(p^1−p^2)=p1^(1−p1^)n1+p2^(1−p2^)n2−−−−−−−−−−−−−−√ when, again, the following conditions are satisfied:

The number of successes and failures in both populations is larger than or equal to 5. We will base this on the number of successes and failures in both samples. Why the samples? If you recall in our discussion for the test of one proportion, this check of conditions used np0 and n(1−p0) where p0 was the assumed null hypothesis population proportion.

However, here where we are comparing two proportions, we do not know what the population proportions are; but we are assuming that they are equal. For instance, the two population proportions could be 0.65 or 0.30, etc. It doesn't matter as we are assuming they are equal. As a result we do not have a fixed population value to use - thus substitute with the sample proportions for each group.

Hypothesis Testing to Compare Two Population Proportions: p1,p2 When we want to check whether two proportions are different or the same, the two-tailed test is appropriate. If we want to see whether it is true that p1≠p2, that can be written as p1−p2≠0.

Note: To check whether it is true that that p1>p2 , that can be written as p1−p2>0. For p1<p2, that can be written as p1−p2<0

.The test statistic is:

Z∗=p^1−p^2p∗(1−p∗)(1n1+1n2)−−−−−−−−−−−−−−−−√ where p∗=x1+x2n1+n2.

Note: In the denominator we assume that the two proportions have the same variances and estimate that by the pooled estimate.

In other words, if the two population proportions are assumed equal, then we combine the results from the two samples into one sample proportion: the total number of success in the two samples divided by the total sample size of the two samples. This is what p∗ is representing. In Minitab, you need to get into options and select "Use pooled estimate of p for test." If you don't think that is reasonable to assume, then don't check the option.

Let p1 and p2 denote the proportions showing an attribute in populations 1 and 2. We wish to compare the null hypothesis

$H_0 : p_1 = p_2$ with any one of the alternatives

$H_{11} : p_1 > p_2, \quad H_{12} : p_1 < p_2, \quad H_{13} : p_1 \neq p_2$

We have seen that the sample proportion p1, say if n is sufficiently large, has the normal distribution with mean p_1 and variance $p_1q_1/n_{1.}$ Similarly $p_2 \sim N(p_2, p_2q_2/n_2)$ so that

$$Z = \frac{p_1 - p_2}{\sqrt{(\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2})}}$$

Example 2

A sample poll of 240 registered voters' from state A shows 60.3% favouring party A while another poll of 200 registered voters taken at the same time from state B shows 53.5%. Using a 0.01 size test, would you conclude that state A has more supporters for party A than state B

$H_0 : P_A = P_B; \quad H_1 : P_A > P_B$

P= (240x0.603+200x0.535)/440=0.572

$\sigma_{P_A - P_B} = 0.0474$

$Z=(0.603-0.535)/0.0474=1.43$

Decision Rule: Reject null hypothesis if $Z>2.33$. We cannot reject null hypothesis.

Summary

In this study session you have learnt about:

1. **Two – Sample Test of Means**

   When you perform a two-sample t- test, you are actually testing whether the difference between the sample means is equal to zero.

2. **When to Use Two-Sample t-Tests**

   You use two-sample t-tests to determine whether two samples are likely or unlikely to have been selected from the same population (i.e., to decide whether two samples differ significantly based on the selected confidence level).Two- sample t-tests are, at times, overused in place of other more appropriate statistical tests because they are very familiar to many researchers.

3. **Two – Sample Test of Proportions – Large Sample**

Confidence Intervals to Estimate the Difference between Two Population Proportions: p1−p2

Point estimate is the difference between the two sample proportions, written as:

p^1−p^2=y1n1−y2n2

The mean of its sampling distribution is p1−p2 and the standard deviation is given by:

p1^(1−p1^)n1+p2^(1−p2^)n2−−−−−−−−−−−−−−−−−−−√

When the observed number of successes and the observed number of failures are greater than or equal to 5 for both populations, then the sampling distribution of p^1−p^2 is approximately normal and we can use z-methods.

Self-Assessment Questions (SAQs) for study session 7

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 7.1 (Testing Learning Outcomes 7.1)

Briefly discuss Two – Sample Test of Means

SAQ 7.2 (Testing Learning Outcomes 7.2)

Highlight on When to Use Two-Sample t-Tests

SAQ 7.3 (Testing Learning Outcomes 7.3)

Discuss on Two – Sample Test of Proportions – Large Sample

References

John, E. F.(1974). Modern Elementary Statistics, London: International Edition. Prentice Hall.

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). Statistical Theory and Methods. Ibadan: Joytal Press.

Study Session 8: Tests of Hypothesis

Introduction

A hypothesis test is a statistical test that is used to determine whether there is enough evidence in a sample of data to infer that a certain condition is true for the entire population. A hypothesis test examines two opposing hypotheses about a population: the null hypothesis and the alternative hypothesis.

This study session presents tests of hypothesis. Null and alternative hypothesis will be discussed. Test Statistic, Critical Region, Significance level and general procedure for testing hypothesis will be discussed as well.

Learning Outcomes for Study Session 8

At the end of this study session, you should be able to:

8.1 Highlight on the Hypothesis

8.2 Explain Hypothesis Testing (P-value approach)

8.3 Discuss Null Hypothesis

8.1 Hypothesis Testing

A statistical hypothesis is a statistical statement, which may or may not be true concerning one or more populations. A test of hypothesis is a rule, which, on the basis of relevant statistic, leads to a decision to accept or reject the null hypothesis.

The rejection of H_0 when it is true is called a Type I error and the acceptance of H_1 when it is false is called a Type II error.

The hypothesis $\theta \leq 20, \theta > 20$ are composite given that $X \sim N(\theta, 25)$ whereas the hypothesis $\theta = 20$ is simple for it completely specifies that the distribution of X is N(20, 25)

8.2 Hypothesis Testing (P-value approach)

You are going to examine the P-Value Approach

P-value approach

The *P*-value approach involves determining "likely" or "unlikely" by determining the probability assuming the null hypothesis were true of observing a more extreme test statistic in the direction of the alternative hypothesis than the one observed. If the *P*-value is small, say less than (or equal to) α, then it is "unlikely." And, if the *P*-value is large, say more than α, then it is "likely."

If the *P*-value is less than (or equal to) α, then the null hypothesis is rejected in favor of the alternative hypothesis. And, if the *P*-value is greater than α, then the null hypothesis is not rejected.

Specifically, the four steps involved in using the *P*-value approach to conducting any hypothesis test are:

1. Specify the null and alternative hypotheses.
2. Using the sample data and assuming the null hypothesis is true, calculate the value of the test statistic. Again, to conduct the hypothesis test for the population mean μ, we use the *t*-statistic $t_{*}=x^{-}{-}\mu s/n\surd$ which follows a *t*-distribution with n - 1 degrees of freedom.

2. Using the known distribution of the test statistic, calculate the ***P*-value**: "If the null hypothesis is true, what is the probability that we'd observe a more extreme test statistic in the direction of the alternative hypothesis than we did?" (Note how this question is equivalent to the question answered in criminal trials:

   "If the defendant is innocent, what is the chance that we'd observe such extreme criminal evidence?")

3. Set the significance level, α, the probability of making a Type I error to be small — 0.01, 0.05, or 0.10. Compare the *P*-value to α. If the *P*-value is less than (or equal to)

α, reject the null hypothesis in favor of the alternative hypothesis. If the *P*-value is greater than α, do not reject the null hypothesis.

In our example concerning the mean grade point average, suppose that our random sample of n = 15 students majoring in mathematics yields a test statistic t^* equaling 2.5. Since n = 15, our test statistic t^* has n - 1 = 14 degrees of freedom. Also, suppose we set our significance level α at 0.05, so that we have only a 5% chance of making a Type I error.

The *P*-value for conducting the **right-tailed** test $H_0 : \mu = 3$ versus $H_A : \mu > 3$ is the probability that we would observe a test statistic greater than $t^* = 2.5$ if the population mean μ really were 3. Recall that probability equals the area under the probability curve. The *P*-value is therefore the area under a $t_{n-1} = t_{14}$ curve and to the *right* of the test statistic $t^* = 2.5$. It can be shown using statistical software that the *P*-value is 0.0127:

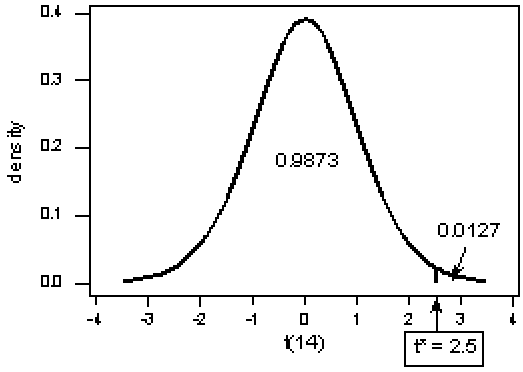


Figure 8.1: P-Value Hypothesis Test 1

The *P*-value, 0.0127, tells us it is "unlikely" that we would observe such an extreme test statistic t^* in the direction of H_A if the null hypothesis were true. Therefore, our initial assumption that the null hypothesis is true must be incorrect. That is, since the *P*-value, 0.0127, is less than α = 0.05, we reject the null hypothesis $H_0 : \mu = 3$ in favor of the alternative hypothesis $H_A : \mu > 3$.

Note that we would not reject $H_0 : \mu = 3$ in favor of $H_A : \mu > 3$ if we lowered our willingness to make a Type I error to α = 0.01 instead, as the *P*-value, 0.0127, is then greater than α = 0.01.

In our example concerning the mean grade point average, suppose that our random sample of *n* = 15 students majoring in mathematics yields a test statistic *t** instead equaling -2.5.

The *P*-value for conducting the **left-tailed** test $H_0 : \mu = 3$ versus $H_A : \mu < 3$ is the probability that we would observe a test statistic less than *t** = -2.5 if the population mean μ really were 3. The *P*-value is therefore the area under a $t_{n-1} = t_{14}$ curve and to the *left* of the test statistic t* = -2.5. It can be shown using statistical software that the *P*-value is 0.0127:

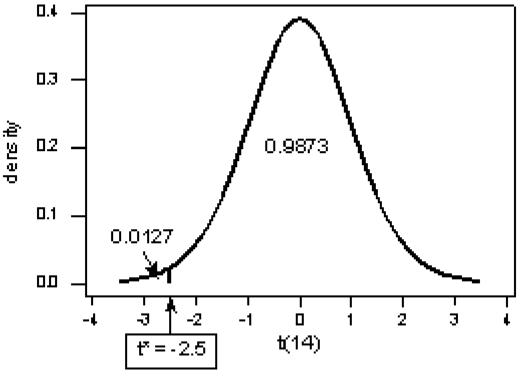


Figure 8.2: P-Value Hypothesis Test 2

The *P*-value, 0.0127, tells us it is "unlikely" that we would observe such an extreme test statistic *t** in the direction of H_A if the null hypothesis were true. Therefore, our initial assumption that the null hypothesis is true must be incorrect. That is, since the *P*-value, 0.0127, is less than α = 0.05, we reject the null hypothesis $H_0 : \mu = 3$ in favor of the alternative hypothesis $H_A : \mu < 3$.

Note that we would not reject $H_0 : \mu = 3$ in favor of $H_A : \mu < 3$ if we lowered our willingness to make a Type I error to α = 0.01 instead, as the *P*-value, 0.0127, is then greater than α = 0.01.

In our example concerning the mean grade point average, suppose again that our random sample of *n* = 15 students majoring in mathematics yields a test statistic *t** instead equaling -2.5. The *P*-value for conducting the **two-tailed** test $H_0 : \mu = 3$ versus $H_A : \mu \neq 3$ is the probability that we would observe a test statistic less than -2.5 or greater than 2.5 if the population mean μ really were 3.

That is, the two-tailed test requires taking into account the possibility that the test statistic could fall into either tail (and hence the name "two-tailed" test). The *P*-value is therefore the area under a $t_{n-1} = t_{14}$ curve to the *left* of -2.5 and to the *right* of the 2.5. It can be shown using statistical software that the *P*-value is 0.0127 + 0.0127, or 0.0254

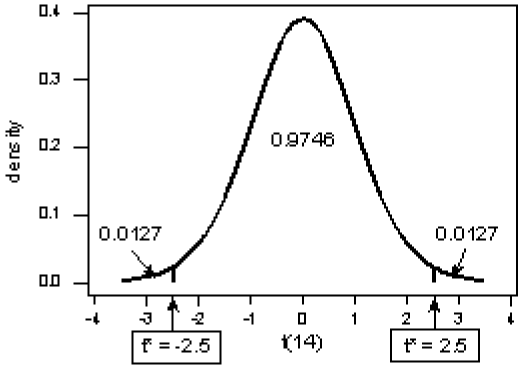


Figure 8.3: P-Value Hypothesis Test 3

Note that the *P*-value for a two-tailed test is always two times the *P*-value for either of the one-tailed tests. The *P*-value, 0.0254, tells us it is "unlikely" that we would observe such an extreme test statistic *t** in the direction of H_A if the null hypothesis were true.

Therefore, our initial assumption that the null hypothesis is true must be incorrect. That is, since the *P*-value, 0.0254, is less than α = 0.05, we reject the null hypothesis $H_0 : \mu = 3$ in favor of the alternative hypothesis $H_A : \mu \neq 3$.

Note that we would not reject $H_0 : \mu = 3$ in favor of $H_A : \mu \neq 3$ if we lowered our willingness to make a Type I error to α = 0.01 instead, as the *P*-value, 0.0254, is then greater than α = 0.01.

Now that we have reviewed the critical value and *P*-value approach procedures for each of three possible hypotheses, let's look at three new examples — one of a right-tailed test, one of a left-tailed test, and one of a two-tailed test.

The good news is that, whenever possible, we will take advantage of the test statistics and *P*-values reported in statistical software, such as Minitab, to conduct our hypothesis tests in this course.

In Text Question

The *P*-value approach involves determining "likely" or "unlikely". **True/False**

In Text Answer

True

8.3 Null Hypothesis

A hypothesis is a speculation or theory based on insufficient evidence that lends itself to further testing and experimentation. With further testing, a hypothesis can usually be proven true or false. Let's look at an example. Little Susie speculates, or hypothesizes, that the flowers she waters with club soda will grow faster than flowers she waters with plain water. She waters each plant daily for a month (experiment) and proves her hypothesis true!

A null hypothesis is a hypothesis that says there is no statistical significance between the two variables in the hypothesis. It is the hypothesis that the researcher is trying to disprove.

In the example, Susie's null hypothesis would be something like this: There is no statistically significant relationship between the type of water I feed the flowers and growth of the flowers. A researcher is challenged by the null hypothesis and usually wants to disprove it, to demonstrate that there is a statistically-significant relationship between the two variables in the hypothesis.

8.3.1 Alternative Hypothesis

An alternative hypothesis simply is the inverse, or opposite, of the null hypothesis. So, if we continue with the above example, the alternative hypothesis would be that there IS indeed a statistically-significant relationship between what type of water the flower plant is fed and growth. More specifically, here would be the null and alternative hypotheses for Susie's study:

Null: If one plant is fed club soda for one month and another plant is fed plain water, there will be no difference in growth between the two plants.

Alternative: If one plant is fed club soda for one month and another plant is fed plain water, the plant that is fed club soda will grow better than the plant that is fed plain water.

The hypothesis, H_0 is called the **Null hypothesis** while H_1 is called the **Alternative hypothesis**. The former (H_0) is a statement or an assumption, which we want to test or which is accepted until the evidence proves otherwise, while the latter (H_1) is a statement, which states what we suppose will happen if the null hypothesis is not correct. Thus, H_1 is usually stated as:

$H_1 : \theta \neq \theta_1$or

$\theta < \theta_0$ or

$\theta > \theta_0$

We partitioned the sample space into two regions c and c^* such that if the sample values fall in c, we reject H_0, but if it falls in c^*, we do otherwise (accept H_0). The region c is called the **critical region** of the rest, and it is defined as the region that corresponds to the rejection of the null hypothesis.

We can make a mistake of accepting H_1, when H_0 is true or accepting H_0 when H_1 is true; such mistakes are called **Type I and II errors** respectively.

Table 8.1: Type I and II

Decision	True	False
Reject H_0	Type I error	Correct decision
Accept H_0	Correct decision	Type II error

8.3.2 Level of Significance (α)

In testing a given hypothesis, the maximum probability with which we would be willing to risk a Type I error is called the level of significance of the test. The value α needs to be specified before any sample is drawn so that results obtained will not influence our choice.

8.3.3 Test Statistic

This is the statistic formula whose value has to be computed from the available sample data and, thus, determines the acceptance or rejection of that particular known hypothesis H_0 under investigation. The text statistic may require Normal, t, X^2 or F approach, as the case may be.

8.3.4 General Procedures for Testing a Statistical Hypothesis

The following steps are summarized as procedures for test of hypothesis:

1. Formulate the Null and Alternate hypothesis.
2. Determine the appropriate test statistic, and compute its value.
3. Choose α, the level of significance.
4. Determine the critical region.
5. Make a statistical decision and
6. Conclude i.e. interpret your result.

Summary

In this study session you have learnt about:

1. **Hypothesis Testing**

   A statistical hypothesis is a statistical statement, which may or may not be true concerning one or more populations. A test of hypothesis is a rule, which, on the basis of relevant statistic, leads to a decision to accept or reject the null hypothesis.

2. **Hypothesis Testing (P-value approach)**
   You are going to examine the P-Value Approach

   The *P*-value approach involves determining "likely" or "unlikely" by determining the probability assuming the null hypothesis were true of observing a more extreme test statistic in the direction of the alternative hypothesis than the one observed.

3. **Null Hypothesis**

   A hypothesis is a speculation or theory based on insufficient evidence that lends itself to further testing and experimentation. With further testing, a hypothesis can usually be proven true or false. Let's look at an example.

   Little Susie speculates, or hypothesizes, that the flowers she waters with club soda will grow faster than flowers she waters with plain water. She waters each plant daily for a month (experiment) and proves her hypothesis true!

Self-Assessment Questions (SAQs) for study session 8

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 8.1 (Testing Learning Outcomes 8.1)

Define Hypothesis Testing

SAQ 8.2 (Testing Learning Outcomes 8.2)

Explain Hypothesis Testing (P-value approach)

SAQ 8.3 (Testing Learning Outcomes 8.3)

Discuss Null Hypothesis

References

John, E. F. (1974). Modern Elementary Statistics, International Edition. London: Prentice Hall. Murray, R. S. (1972)Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). Statistical Theory and Methods. Ibadan: Joytal: Press.

Study Session 9: One Sample Test of Mean and Proportion – Large Sample

Introduction

In estimation your focused will be on explicitly on techniques for one and two samples and talk on estimation for a specific parameter (e.g., the mean or proportion of a population), for differences (e.g., difference in means, the risk difference) and ratios (e.g., the relative risk and odds ratio). Here we will focus on procedures for one and two samples when the outcome is either continuous (and we focus on means) or dichotomous (and we focus on proportions).

The aim of this study session is to introduce you to one test of mean and proportion for large sample. You shall also work examples on large sample test of mean and proportion.

Learning Outcomes for Study Session 9

At the end of this study session, you should be able to:

9.1 One and two sided tests of significance

9.2 One – Sample Test of Mean – Large Sample

9.1 One and two sided tests of significance

When we use a test of significance to compare two groups we usually start with the null hypothesis that there is no difference between the populations from which the data come. If this hypothesis is not true the alternative hypothesis must be true - that there is a difference.

Since the null hypothesis specifies no direction for the difference nor does the alternative hypothesis, and so we have a two sided test. In a one sided test the alternative hypothesis does specify a direction - for example, that an active treatment is better than a placebo.

This is sometimes justified by saying that we are not interested in the possibility that the active treatment is worse than no treatment. This possibility is still part of the test; it is part of the null hypothesis, which now states that the difference in the population is zero or in favour of the placebo.

A one sided test is sometimes appropriate.

Box 9.1: Luthra et al Investigated:

The effects of laparoscopy and hydrotubation on the fertility of women presenting at an infertility clinic.1 After some months laparoscopy was carried out on those who had still not conceived. These women were then observed for several further months and some of these women also conceived.

The conception rate in the period before laparoscopy was compared with that afterwards. The less fertile a woman is the longer it is likely to take her to conceive. Hence, the women who had the laparoscopy should have a lower conception rate (by an unknown amount) than the larger group who entered the study, because the more fertile women had conceived before their turn for laparoscopy came.

To see whether laparoscopy increased fertility, Luthra et al tested the null hypothesis that the conception rate after laparoscopy was less than or equal to that before. The alternative hypothesis was that the conception rate after laparoscopy was higher than that before. A two sided test was inappropriate because if the laparoscopy had no effect on fertility the conception rate after laparoscopy was expected to be lower.

One sided tests are not often used, and sometimes they are not justified. Consider the following example. Twenty five patients with breast cancer were given radiotherapy treatment of 50 Gy in fractions of 2 Gy over 5 weeks. 2 Lung function was measured initially, at one week, at three months, and at one year. The aim of the study was to see whether lung function was lowered following radiotherapy.

In Text Question

One sided tests are not often used, and sometimes they are not justified. **True/False**

In Text Answer

True

Some of the results are shown in the table, the forced vital capacity being compared between the initial and each subsequent visit using one sided tests. The direction of the one sided tests was not specified, but it may appear reasonable to test the alternative hypothesis that forced vital capacity decreases after radiotherapy, as there is no reason to suppose that damage to the lungs would increase it.

The null hypothesis is that forced vital capacity does not change or increases. If the forced vital capacity increases, this is consistent with the null hypothesis, and the more it increases the more consistent the data are with the null hypothesis. Because the differences are not all in the same direction, at least one P value should be greater than 0.5.

What has been done here is to test the null hypothesis that forced vital capacity does not change or decreases from visit 1 to visit 2 (nine week), and to test the null hypothesis that it does not change or increases from visit 1 to visit 3 (three months) or visit 4 (one year). These authors seem to have carried out one sided tests in both directions for each visit and then taken the smaller probability.

If there is no difference in the population the probability of getting a significant difference by this approach is 10%, not 5% as it should be. The chance of a spurious significant difference is doubled. Two sided tests should be used, which would give probabilities of 0.26, 0.064, and 0.38, and no significant differences.

In general a one sided test is appropriate when a large difference in one direction would lead to the same action as no difference at all. Expectation of a difference in a particular direction is not adequate justification. In medicine, things do not always work out as expected, and researchers may be surprised by their results.

For example 1:

Galloe et al found that oral magnesium significantly increased the risk of cardiac events, rather than decreasing it as they had hoped.3 If a new treatment kills a lot of patients we should not simply abandon it; we should ask why this happened.

Two sided tests should be used unless there is a very good reason for doing otherwise. If one sided tests are to be used the direction of the test must be specified in advance. One sided tests should never be used simply as a device to make a conventionally non-significant difference significant.

Example 2

Let us consider a test of whether a coin is symmetric or not and consider the particular hypotheses.

1. $H_{01} : p = \frac{1}{2}, H_{11} : p < \frac{1}{2}$ - One tail test lower tail

2. $H_{02} : p = \frac{1}{2}, H_{12} : p > \frac{1}{2}$ - One tail test Upper tail

3. $H_{03} : p = \frac{1}{2}, H_{13} : p \neq \frac{1}{2}$. Two tail test

where p is the probability that the coin shows a head.With the coin in the dock accused of being biased, we call for evidence, which is the number of heads in 10 heads of the coin.In the first case, one would be inclined to reject H_0 if X is equal to, say, 0, 1, or 2 and to accept H_0 for other values of X.

That is only values of X which are extremely less than 5 will lead to the rejection of H_0.In the second case, one would be inclined to reject H_0 if X takes, say, any of the values 10, 9, 8, i.e. only values of X, which are extremely larger than 5 will lead to the rejection of H_0.In the third case, values of X, which are either extremely larger than or extremely less than 5 seem to qualify for the critical region.

9.2 One – Sample Test of Mean – Large Sample

A hypothesis about a population mean μ may be tested by obtaining the mean of a random sample of size n from the population. Suppose a random sample of size n_A from population A yields a mean $\overline{X}_A$; we know that provided n_A is large and sampling is done from a finite population and it is done with replacement then $\bar{X}_A \sim N(\mu_A, \sigma_A^2 / n_A)$ and the test statistic is $Z = \dfrac{\bar{X}_A - \mu_A}{\sqrt{\dfrac{\sigma_A^2}{n_A}}}$ which has the standard normal distribution; μ and σ being population mean and standard deviation respectively. When the population variance σ^2 is not known but n is sufficiently large, we may use $\hat{s}^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ in its place. The approximation is still good.

Example 1

The mean weight of 64 men is 63kg. The mean weight for the population is 60kg, with a standard deviation of 12kg. Is the mean weight of the sample greater than the population mean weight at the 5% level?

Solution

Set up the hypothesis

H_o : Mean Weight = 60kg

H_1 : Mean Weight $\neq$ 60kg

$$\mathrm{Se}(\bar{x}) = \frac{12}{\sqrt{64}} = \frac{12}{8} = 1.5 \ kilos$$

$$Z = \frac{\bar{x} - \mu}{S.E(\bar{x})}$$

$$= \frac{63-60}{1.5} = \frac{3}{1.5} \quad = 2.0$$

Comparing the calculated Z statistic at the 5% level tabulated Z value = 1.96

Decision: Since the calculated Z value of 2.0 lie outside -1.96 and +1.96 table value of Z at 5% level of significance, we reject the null hypothesis (H_0) and conclude that the mean weight of the sample is not 60kg. The result is said to be significant at the 5% level.

9.2.1 One Sample Test- Proportion- Large Sample

Example 2

In a recent poll 0.65 of the 91 registered voters polled favoured party A, whereas the proportion that favoured the party at the last general election before the poll was 0.575.

Would this be a justification for claiming an improved popularity for party A at the 0.05 level of significance?

Solution

$H_0 : P = p_0 = 0.575$

$H_1 : P > 0.575$

$$= \frac{(0.65 - 0.575)}{\sqrt{\frac{(0.575 \times 0.425}{91}}} = 1.45$$

The critical level for a test of size 0.05 is for an upper tail, 1.645. Hence we cannot reject H_0

Summary

In this study session you have learnt about:

1. One and two sided tests of significance

When we use a test of significance to compare two groups we usually start with the null hypothesis that there is no difference between the populations from which the data come. If this hypothesis is not true the alternative hypothesis must be true - that there is a difference.

2. One – Sample Test of Mean – Large Sample

A hypothesis about a population mean μ may be tested by obtaining the mean of a random sample of size n from the population. Suppose a random sample of size n$_A$ from population A yields a mean $\overline{X}_A$; we know that provided n$_A$ is large and sampling is done from a finite population and it is done with replacement then $\overline{X}_A \sim N(\mu_A, \sigma_A^2 / n_A)$ and the test statistic is $Z = \frac{\overline{X}_A - \mu_A}{\sqrt{\frac{\sigma_A^2}{n_A}}}$ which has the standard normal distribution; μ and σ being population mean and standard deviation respectively. When the population variance σ^2 is not known but n is sufficiently large, we may use $\hat{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ in its place. The approximation is still good.

Self-Assessment Questions (SAQs) for study session 9

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 9.1 (Testing Learning Outcomes 9.1)

Briefly explain One and two sided tests of significance

SAQ 9.2 (Testing Learning Outcomes 9.2)

Discuss One Sample Test of Mean – Large Sample

References

John, E. F.(1974). *Modern Elementary Statistics*, International Edition. London: Prentice Hall.

Murray, R. S. (1972) *Schaum's Outline Series. Theory and Problems of Statistics*. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). *Statistical Theory and Methods*. Ibadan: Joytal Press.

Lund MB, Myhre KI, Melsom H, Johansen B The effect on pulmonary function of tangential field technique in radiotherapy for carcinoma of the breast. Br J Radiol 1991;64:520–3.

Luthra P, Bland JM, Stanton SL Incidence of pregnancy after laparoscopy and hydrotubation. BMJ 1982;284:1013.

Galloe AM, Rasmussen HS, Jorgensen LN, Aurup P, Balslov S, Cintin C, Graudal N, McNair P

Influence of oral magnesium supplementation on cardiac events among survivors of an acute myocardial infarction. BMJ 1993;307:585–7.

Study Session 10: Two Sample Test of Means and Proportions – Large Sample

Introduction

The process of hypothesis testing involves setting up two competing hypotheses, the null hypothesis and the alternate hypothesis. One selects a random sample (or multiple samples when there are more comparison groups), computes summary statistics and then assesses the likelihood that the sample data support the research or alternative hypothesis.

Similar to estimation, the process of hypothesis testing is based on probability theory and the Central Limit Theorem.

Learning Outcomes for Study Session 10

At the end of this study session, you should be able to:

10.1 Explain Two-Sample *t*-Test for Equal Means

10.2 Discuss Test Statistics for Testing

10.1 Two-Sample *t*-Test for Equal Means

Purpose Test if two population means are equal:

The two-sample *t*-test is used to determine if two population means are equal. A common application is to test if a new process or treatment is superior to a current process or treatment.

There are several variations on this test:

1. The data may either be paired or not paired. By paired, we mean that there is a one-to-one correspondence between the values in the two samples. That is, if X_1, X_2, ..., X_n and Y_1, Y_2, ... , Y_n are the two samples, then X_i corresponds to Y_i. For paired samples, the difference X_i - Y_i is usually calculated.

   For unpaired samples, the sample sizes for the two samples may or may not be equal. The formulas for paired data are somewhat simpler than the formulas for unpaired data.

2. The variances of the two samples may be assumed to be equal or unequal. Equal variances yields somewhat simpler formulas, although with computers this is no longer a significant issue.
3. In some applications, you may want to adopt a new process or treatment only if it exceeds the current treatment by some threshold. In this case, we can state the null hypothesis in the form that the difference between the two populations means is equal to some constant $\mu 1-\mu 2=d0$ where the constant is the desired threshold.

Definition The two-sample *t*-test for unpaired data is defined as:

H_0: $\mu 1=\mu 2$

H_a: $\mu 1\neq\mu 2$

Test Statistic: $T=Y1^{¯}-Y2^{¯}s21/N1+s22/N2\surd$ where N_1 and N_2 are the sample sizes, $Y1^{¯}$ and $Y2^{¯}$ are the sample means, and $s21$ and $s22$ are the sample variances. If equal variances are assumed, then the formula reduces to: $T=Y1^{¯}-Y2^{¯}sp1/N1+1/N2\surd$ where $s2p=(N1-1)s21+(N2-1)s22N1+N2-2$

Significance **Level:** α.

Critical Region: Reject the null hypothesis that the two means are equal if

$|T| > t_{1-\alpha/2,v}$ where $t_{1-\alpha/2,v}$ is the critical value of the *t* distribution with *v* degrees of freedom where

$\upsilon=(s21/N1+s22/N2)2(s21/N1)2/(N1-1)+(s22/N2)2/(N2-1)$

If equal variances are assumed, then $\nu = N_1 + N_2 - 2$

Two Sample t- Test Example

The following two-sample *t*-test was generated for the auto83b.dat data set. The data set contains miles per gallon for U.S. cars (sample 1) and for Japanese cars (sample 2); the summary statistics for each sample are shown below.

SAMPLE 1: number of observations = 249 MEAN = 20.14458

Standard Deviation = 6.41470 Standard Error Of The Mean = 0.40652

SAMPLE 2:

Number Of Observations = 79

Mean = 30.48101

Standard Deviation = 6.10771

Standard Error Of The Mean = 0.68717 We are testing the hypothesis that the population means are equal for the two samples. We assume that the variances for the two samples are equal.

H_0: $\mu_1 = \mu_2$

H_a: $\mu_1 \neq \mu_2$ Test statistic: T = -12.62059 Pooled standard deviation: s_p = 6.34260

Degrees of freedom: $v = 326$

Significance level: $\alpha = 0.05$ Critical value (upper tail): $t_{1-\alpha/2,v} = 1.9673$

Critical region: Reject H_0 if $|T| > 1.9673$

The absolute value of the test statistic for our example, 12.62059, is greater than the critical value of 1.9673, so we reject the null hypothesis and conclude that the two population means are different at the 0.05 significance level.

In general, there are three possible alternative hypotheses and rejection regions for the one-sample *t*-test:

Alternative Hypothesis	**Rejection Region**
H_a: $\mu_1 \neq \mu_2$	$|T| > t_{1-\alpha/2,\nu}$
H_a: $\mu_1 > \mu_2$	$T > t_{1-\alpha,\nu}$
H_a: $\mu_1 < \mu_2$	$T < t_{\alpha,\nu}$

Figure 10.1: Hypotheses and rejection regions for the one-sample *t*-test:

For our two-tailed *t*-test, the critical value is $t_{1-\alpha/2,v} = 1.9673$, where $\alpha = 0.05$ and $v = 326$. If we were to perform an upper, one-tailed test, the critical value would be $t_{1-\alpha,v}$ = 1.6495. The rejection regions for three possible alternative hypotheses using our example data are shown below.

If a random sample of size n$_x$ taken from an X-population yields mean $\overline{X}$ and an independent one of size n$_Y$ from a Y-population yields mean $\overline{Y}$,we know that the sampling distribution of the difference $\overline{X}-\overline{Y}$ is normal with mean $\mu_{\overline{X}-\overline{Y}} = \mu_X - \mu_Y$ the difference of the population means and variance $\sigma^2_{\overline{X}-\overline{Y}} = \sigma^2_{\overline{X}} + \sigma^2_{\overline{Y}} = \sigma^2_X/n_X + \sigma^2_Y/n_Y$ where σ^2_Y is the variance of the Y-population. i.e. $\overline{X}-\overline{Y} \sim N\left(\mu_X - \mu_Y, \sigma^2_X/n_X + \sigma^2_Y/n_Y\right)$.

$$Z = \frac{\overline{X}-\overline{Y}-(\mu_X-\mu_Y)}{\sqrt{\left(\sigma^2_X/n_X + \sigma^2_Y/n_Y\right)}} \sim N(0,1)$$

For testing the null hypothesis μ$_X$ – μ$_Y$ = 0 or μ$_X$ = μ$_Y$ against the alternative H$_1$: μ$_X$ ≠ μ$_Y$, say, the decision rule, is, for an α-size two-tail test:

Reject H$_0$ if Z lies outside $(z_{\alpha/2},\ z_{1-\alpha/2})$:

Otherwise, do not reject it.

Observe that $z = \frac{(\overline{X}-\overline{Y})}{\sqrt{\left(\sigma^2_X/n_X + \sigma^2_Y/n_Y\right)}}$ when H$_0$ holds, and it is assumed that σ^2_X and σ^2_Y are known.

When σ^2_X and σ^2_Y are not known, provided n$_X$ and n$_Y$ are sufficiently large, they may be estimated by

$$\hat{s}^2_X = \frac{1}{n_X}\sum_{i=1}^{n_X}(X-\overline{X})^2 \quad , \quad \hat{s}^2_Y = \frac{1}{n_Y}\sum_{i=1}^{n_Y}(X-\overline{X})^2$$

Example 1

The mean weight of 50 pigs raised on diet A is 68.2 k9, while that of 60 pigs raised on diet B is 67.0 kg. Test the hypothesis that Diet A is superior to diet B if is known that the population variance of the weights of pigs raised on A is 12.25 kg^2 while that of pigs raised on B is 15.21 kg^2.

Solution

Let μ_A and μ_B denote the population means of weights of pigs raised on A and B respectively. The null hypothesis, then is

$H_0 : \mu_A = \mu_B$

while the alternative is

$H_1 : \mu_A > \mu_B$ which calls for an upper-tail test.

$$\sigma_{\bar{X}_A - \bar{X}_B} = \frac{12.25}{50} + \frac{15.21}{60} = 0.4985$$

$$Z = \frac{(68.2 - 67.0)}{0.706} = 1.7$$

Decision Rule: Reject H_0 if $Z \geq 1.645$ for a 0.05 level of significance test.

We reject H_0 on the basis of the evidence provided by the sample.

10.1.1 Tests with Two Independent Samples

There are many applications where it is of interest to compare two independent groups with respect to their mean scores on a continuous outcome. Here we compare means between groups, but rather than generating an estimate of the difference, we will test whether the observed difference (increase, decrease or difference) is statistically significant or not.

Remember, that hypothesis testing gives an assessment of statistical significance, whereas estimation gives an estimate of effect and both are important.

Here you discuss the comparison of means when the two comparison groups are independent or physically separate. The two groups might be determined by a particular attribute (e.g., sex, diagnosis of cardiovascular disease) or might be set up by the investigator (e.g., participants assigned to receive an experimental treatment or placebo).

The first step in the analysis involves computing descriptive statistics on each of the two samples. Specifically, we compute the sample size, mean and standard deviation in each sample and we denote these summary statistics as follows:

for sample 1:

- n1
- $\bar{X}_1$
- s1

for sample 2:

- n2

- $\bar{X}_2$
- s2

The designation of sample 1 and sample 2 is arbitrary. In a clinical trial setting the convention is to call the treatment group 1 and the control group 2. However, when comparing men and women, for example, either group can be 1 or 2.

In the two independent samples application with a continuous outcome, the parameter of interest in the test of hypothesis is the difference in population means, μ_1-μ_2. The null hypothesis is always that there is no difference between groups with respect to means, i.e.,

$$H_0: \mu_1 - \mu_2 = 0$$

The null hypothesis can also be written as follows: H_0: $\mu_1 = \mu_2$. In the research hypothesis, an investigator can hypothesize that the first mean is larger than the second (H_1: $\mu_1 > \mu_2$), that the first mean is smaller than the second (H_1: $\mu_1 < \mu_2$), or that the means are different (H_1: $\mu_1 \neq \mu_2$). The three different alternatives represent upper-, lower-, and two-tailed tests, respectively. The following test statistics are used to test these hypotheses.

In Text Question

The designation of sample 1 and sample 2 is arbitrary. In a clinical trial setting the convention is to call the treatment group 1 and the control group 2.**True/False**

In Text Answer

True

10.2 Test Statistics for Testing

The following are test of statistics of two sample mean of proportions:

H_0: $\mu_1 = \mu_2$

- if $n_1 \geq 30$ and $n_2 \geq 30$
  - $z = \dfrac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$
- if $n_1 < 30$ or $n_2 < 30$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$ where df $=n_1+n_2-2$.

The formulas above assume equal variability in the two populations (i.e., the population variances are equal, or $s_1^2 = s_2^2$). This means that the outcome is equally variable in each of the comparison populations. For analysis, we have samples from each of the comparison populations. If the sample variances are similar, then the assumption about variability in the populations is probably reasonable.

As a guideline, if the ratio of the sample variances, s_1^2/s_2^2 is between 0.5 and 2 (i.e., if one variance is no more than double the other), then the formulas above are appropriate. If the ratio of the sample variances is greater than 2 or less than 0.5 then alternative formulas must be used to account for the heterogeneity in variances.

The test statistics include Sp, which is the pooled estimate of the common standard deviation (again assuming that the variances in the populations are similar) computed as the weighted average of the standard deviations in the samples as follows:

$$S_p = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$$

Because we are assuming equal variances between groups, we pool the information on variability (sample variances) to generate an estimate of the variability in the population. Note: Because Sp is a weighted average of the standard deviations in the sample, Sp will always be in between s_1 and s_2.)

Example 1

Data measured on n=3,539 participants who attended the seventh examination of the Offspring in the Framingham Heart Study are shown below.

	Men			**Women**		
Characteristic	N	$\bar{X}$	S	n	$\bar{X}$	s
Systolic Blood Pressure	1,623	128.2	17.5	1,911	126.5	20.1
Diastolic Blood Pressure	1,622	75.6	9.8	1,910	72.6	9.7

Total Serum Cholesterol	1,544	192.4	35.2	1,766	207.1	36.7
Weight	1,612	194.0	33.8	1,894	157.7	34.6
Height	1,545	68.9	2.7	1,781	63.4	2.5
Body Mass Index	1,545	28.8	4.6	1,781	27.6	5.9

Suppose we now wish to assess whether there is a statistically significant difference in mean systolic blood pressures between men and women using a 5% level of significance.

- **Step 1.** Set up hypotheses and determine level of significance

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ $\alpha=0.05$

- **Step 2.** Select the appropriate test statistic.

Because both samples are large ($\geq$ 30), we can use the Z test statistic as opposed to t. Note that statistical computing packages use t throughout. Before implementing the formula, we first check whether the assumption of equality of population variances is reasonable.

The guideline suggests investigating the ratio of the sample variances, s_1^2/s_2^2. Suppose we call the men group 1 and the women group 2. Again, this is arbitrary; it only needs to be noted when interpreting the results. The ratio of the sample variances is $17.5^2/20.1^2 = 0.76$, which falls between 0.5 and 2 suggesting that the assumption of equality of population variances is reasonable. The appropriate test statistic is

$$z = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} .$$

- **Step 3.** Set up decision rule.

This is a two-tailed test, using a Z statistic and a 5% level of significance. Reject H_0 if $Z \leq -1.960$ or is $Z \geq 1.960$.

- **Step 4.** Compute the test statistic.

We now substitute the sample data into the formula for the test statistic identified in Step 2. Before substituting, we will first compute Sp, the pooled estimate of the common standard deviation.

$$S_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$S_p = \sqrt{\frac{(1623-1)\,17.5^2 + (1911-10)\,20.1^2}{1623+1911-2}} = \sqrt{\mathbf{359.12}} = \mathbf{19.0}$$

Notice that the pooled estimate of the common standard deviation, Sp, falls in between the standard deviations in the comparison groups (i.e., 17.5 and 20.1). Sp is slightly closer in value to the standard deviation in the women (20.1) as there were slightly more women in the sample. Recall, Sp is a weight average of the standard deviations in the comparison groups, weighted by the respective sample sizes.

Now the test statistic:

$$Z = \frac{128.2 - 126.5}{19.0\sqrt{\frac{1}{162.3} + \frac{1}{1911}}} = \frac{1.7}{0.64} = \mathbf{2.66}$$

- **Step 5.** Conclusion.

We reject H_0 because 2.66 $\geq$ 1.960. We have statistically significant evidence at α=0.05 to show that there is a difference in mean systolic blood pressures between men and women. The p-value is $p < 0.010$.

Here again we find that there is a statistically significant difference in mean systolic blood pressures between men and women at $p < 0.010$. Notice that there is a very small difference in the sample means (128.2-126.5 = 1.7 units), but this difference is beyond what would be expected by chance. Is this a clinically meaningful difference? The large sample size in this example is driving the statistical significance.

A 95% confidence interval for the difference in mean systolic blood pressures is: 1.7 ± 1.26 or (0.44, 2.96). The confidence interval provides an assessment of the magnitude of the difference between means whereas the test of hypothesis and p-value provide an assessment of the statistical significance of the difference.

Above we performed a study to evaluate a new drug designed to lower total cholesterol. The study involved one sample of patients, each patient took the new drug for 6 weeks and had their cholesterol measured. As a means of evaluating the efficacy of the new drug, the mean total cholesterol following 6 weeks of treatment was compared to the NCHS-reported mean total cholesterol level in 2002 for all adults of 203.

At the end of the example, we discussed the appropriateness of the fixed comparator as well as an alternative study design to evaluate the effect of the new drug involving two treatment groups, where one group receives the new drug and the other does not. Here, we revisit the example with a concurrent or parallel control group, which is very typical in randomized controlled trials or clinical trials.

Example 2

A new drug is proposed to lower total cholesterol. A randomized controlled trial is designed to evaluate the efficacy of the medication in lowering cholesterol. Thirty participants are enrolled in the trial and are randomly assigned to receive either the new drug or a placebo.

The participants do not know which treatment they are assigned. Each participant is asked to take the assigned treatment for 6 weeks. At the end of 6 weeks, each patient's total cholesterol level is measured and the sample statistics are as follows.

Table 10.1 : Treatment Table

Treatment	**Sample Size**	**Mean**	**Standard Deviation**
New Drug	15	195.9	28.7
Placebo	15	217.4	30.3

Is there statistical evidence of a reduction in mean total cholesterol in patients taking the new drug for 6 weeks as compared to participants taking placebo? We will run the test using the five-step approach.

- **Step 1.** Set up hypotheses and determine level of significance

H_0: $\mu_1 = \mu_2$ H_1: $\mu_1 < \mu_2$ $\alpha=0.05$

- **Step 2.** Select the appropriate test statistic.

Because both samples are small (< 30), we use the t test statistic. Before implementing the formula, we first check whether the assumption of equality of population variances is reasonable. The ratio of the sample variances, $s_1^2/s_2^2 = 28.7^2/30.3^2 = 0.90$, which falls between 0.5 and 2, suggesting that the assumption of equality of population variances is reasonable. The appropriate test statistic is:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} .$$

- **Step 3.** Set up decision rule.

This is a lower-tailed test, using a t statistic and a 5% level of significance. The appropriate critical value can be found in the t Table (in More Resources to the right). In order to determine the critical value of t we need degrees of freedom, df, defined as $df=n_1+n_2-2 = 15+15-2=28$. The critical value for a lower tailed test with df=28 and $\alpha=0.05$ is -2.048 and the decision rule is: Reject H_0 if $t \leq -2.048$.

- **Step 4.** Compute the test statistic.

We now substitute the sample data into the formula for the test statistic identified in Step 2. Before substituting, we will first compute Sp, the pooled estimate of the common standard deviation.

$$S_p = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$$

$$S_p = \sqrt{\frac{(15-1)\,28.7^2+(15-1)\,30.3^2}{15+15-2}} = \sqrt{870.89} = 29.5$$

Now the test statistic,

$$t = \frac{195.9 - 227.4}{29.5\sqrt{\frac{1}{15}+\frac{1}{15}}} = \frac{-31.5}{10.77} = -\mathbf{2.92}$$

- **Step 5.** Conclusion.

We reject H_0 because -2.92 $\leq$ -2.048. We have statistically significant evidence at α=0.05 to show that the mean total cholesterol level is lower in patients taking the new drug for 6 weeks as compared to patients taking placebo, p < 0.005.

The clinical trial in this example finds a statistically significant reduction in total cholesterol, whereas in the previous example where we had a historical control (as opposed to a parallel control group) we did not demonstrate efficacy of the new drug.

Notice that the mean total cholesterol level in patients taking placebo is 217.4 which is very different from the mean cholesterol reported among all Americans in 2002 of 203 and used as the comparator in the prior example. The historical control value may not have been the most appropriate comparator as cholesterol levels have been increasing over time. In the next section, we present another design that can be used to assess the efficacy of the new drug.

10.2.1 Two – Sample Test of Proportions – Large Sample

Let p1 and p2 denote the proportions showing an attribute in populations 1 and 2. We wish to compare the null hypothesis

$H_0 : p_1 = p_2$ with any one of the alternatives

$$H_{11} : p_1 > p_2, \quad H_{12} : p_1 < p_2, \quad H_{13} : p_1 \neq p_2$$

We have seen that the sample proportion p1, say if n is sufficiently large, has the normal distribution with mean p_1 and variance p_1q_1/n_1. Similarly $p_2 \sim N(p_2, p_2q_2/n_2)$ so that

$$Z = \frac{p_1 - p_2}{\sqrt{(\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2})}}$$

Example 1

A sample poll of 240 registered voters' from state A shows 60.3% favouring party A while another poll of 200 registered voters taken at the same time from state B shows 53.5%. Using a 0.01 size test, would you conclude that state A has more supporters for party A than state B

$H_0 : P_A = P_B; \quad H_1 : P_A > P_B$

P= (240x0.603+200x0.535)/440=0.572

$\sigma_{P_A - P_B}$ = 0.0474

Z=(0.603-0.535)/0.0474=1.43

Decision Rule: Reject null hypothesis if Z>2.33. We cannot reject null hypothesis.

Summary

In this study session you have learnt about:

(1) Two-Sample *t*-Test for Equal Means

The two-sample *t*-test is used to determine if two population means are equal. A common application is to test if a new process or treatment is superior to a current process or treatment.

There are several variations on this test:

1. The data may either be paired or not paired. By paired, we mean that there is a one-to-one correspondence between the values in the two samples. That is, if X_1, X_2, ..., X_n and Y_1, Y_2, ... , Y_n are the two samples, then X_i corresponds to Y_i. For paired samples, the difference X_i - Y_i is usually calculated.

   For unpaired samples, the sample sizes for the two samples may or may not be equal. The formulas for paired data are somewhat simpler than the formulas for unpaired data.

2. The variances of the two samples may be assumed to be equal or unequal. Equal variance yields somewhat simpler formulas, although with computers this is no longer a significant issue.

3. In some applications, you may want to adopt a new process or treatment only if it exceeds the current treatment by some threshold. In this case, we can state the null hypothesis in the form that the difference between the two populations means is equal to some constant $\mu 1 - \mu 2 = d0$ where the constant is the desired threshold.

(2) Test Statistics for Testing

The following are test of statistics of two sample mean of proportions:

H_0: $\mu_1 = \mu_2$

- if $n_1 \geq 30$ and $n_2 \geq 30$
  - $$z = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$
- if $n_1 < 30$ or $n_2 < 30$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$ where df $=n_1+n_2-2$.

The formulas above assume equal variability in the two populations (i.e., the population variances are equal, or $s_1^2 = s_2^2$). This means that the outcome is equally variable in each of the comparison populations. For analysis, we have samples from each of the comparison populations. If the sample variances are similar, then the assumption about variability in the populations is probably reasonable.

Self-Assessment Questions (SAQs) for study session 10

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 10.1 (Testing Learning Outcomes 10.1)

Lst Two-Sample *t*-Test for Equal Means

SAQ 10.2 (Testing Learning Outcomes 10.2)

Explain Test Statistics for Testing

References

John, E. F.(1974). *Modern Elementary Statistics*, International Edition. London: Prentice Hall.

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). *Fundamentals of Statistics* Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). Statistical Theory and Methods. Ibadan: Joytal Press.

Study Session 11: Regression Analysis

Introduction

Regression analysis is a statistical tool for the investigation of relationships between variables. Usually, the investigator seeks to ascertain the causal effect of one variable upon another the effect of a price increase upon demand, for example, or the effect of changes in the money supply upon the inflation rate.

To explore such issues, the investigator assembles data on the underlying variables of interest and employs regression to estimate the quantitative effect of the causal variables upon the variable that they influence. The investigator also typically assesses the "statistical significant" of the estimated relationships, that is, the degree of confidence that the true relationship is close to the estimated relationship.

In this study session, you shall consider situations where two variables are observed on each unit such data are, for instance, age, intelligence quotient data form, a bivariate data etc. Scatter diagram will be discussed as well as least square line. We will fit least square line to using a bivariate data.

Learning Outcomes for Study Session 11

At the end of this study session, you should be able to:

11.1Explain Scatter Diagram

11.2 Discuss Least Square

11.1 Scatter Diagram

Scatter graph method is a graphical technique of separating fixed and variable components of mixed cost by plotting activity level along x-axis and corresponding total cost (mixed cost) along y-axis. A regression line is then drawn on the graph by visual inspection.

The scatter diagram is known by many names, such as scatter plot, scatter graph, and correlation chart. This diagram is drawn with two variables, usually the first variable is independent and the second variable is dependent on the first variable.

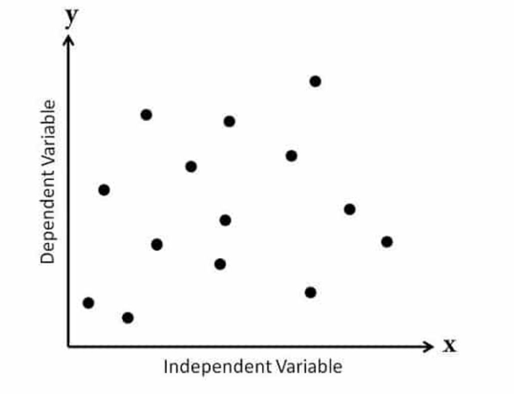


Figure 11.1: Scatter Diagram

The scatter diagram is used to find the correlation between these two variables. This diagram shows you how closely the two variables are related. After determining the correlation between the variables, you can easily predict the behavior of the other variable. This chart is very useful when one variable is easy to measure and the other is not.

For example, let’s say that you are analyzing the pattern of accidents on a highway. You select the two variables motor speed and number of accidents, and draw the diagram.

Once the diagram is completed, you notice that as the speed of vehicle increases, the number of accidents also goes up. This shows that there is a relation between the speed of vehicles and accidents happening on the highway.

11.1.1 Type of Scatter Diagram

The scatter diagram can be categorized into several types of classifications; however, here I will discuss two types of classifications that will cover most types of scatter diagrams. The first classification is based on the type of correlation, and the second classification is based on the slope of trend.

I am giving you two types of classifications because it will show you the same chart with two different perspectives which will build a solid understanding for you regarding the scatter diagram.

According to the type of correlation, scatter diagrams can be divided into following categories:

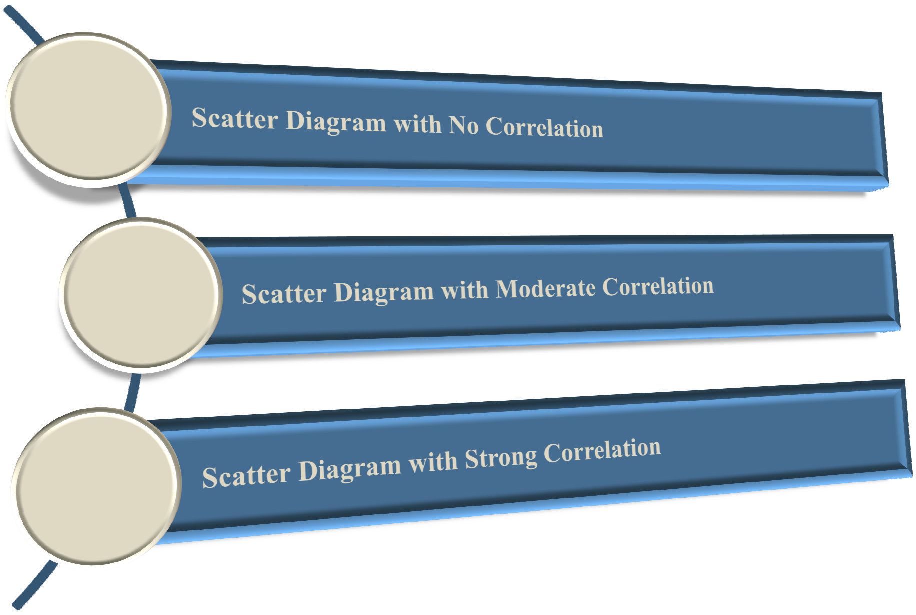

Scatter Diagram with No Correlation
Scatter Diagram with Moderate Correlation
Scatter Diagram with Strong Correlation

Figure 11.2: Types of Scatter Diagram

The scatter diagram is “A correlation chart that uses a regression line to explain or to predict how the change in an independent variable will change a dependent variable.”

Scatter diagram helps you see the changes in the dependent variable if you make any change to the independent variable. Since this diagram shows you the correlation between the variables, it is also known as a correlation chart.

Usually independent variable is plotted along the horizontal axis (x-axis) and dependent variable is plotted on the vertical axis (y-axis). The independent variable is also known as the control parameter because it influences the behavior of the dependent variable.

It is not necessary for one parameter to be a controlling parameter. You can draw the scatter diagram with both variables independent to each other. In this case you can draw any variable on any axis.

Please note that the scatter diagram is different than the Ishikawa or fishbone diagram. With the Ishikawa diagram you see the effect of a cause, and in the scatter diagram you analyze the relationship between the two variables.

11.1.2 Scatter Diagram with No Correlation

This type of diagram is also known as "Scatter Diagram with Zero Degree of Correlation".

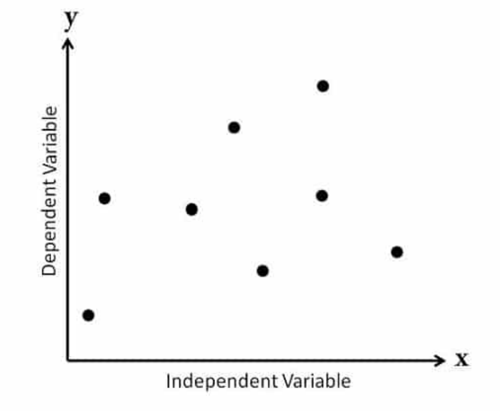


Figure 11.3: Scatter Diagram with No Correlation

In this type of scatter diagram, data points are spread so randomly that you cannot draw any line through them. In this case you can say that there is no relation between these two variables.

11.1.3 Scatter Diagram with Moderate Correlation

This type of diagram is also known as "Scatter Diagram with Low Degree of Correlation".

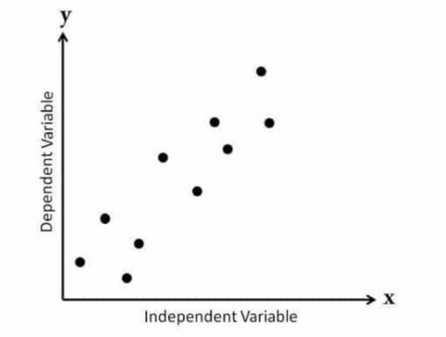


Figure 11.4: Scatter Diagram with Moderate Correlation

Here, the data points are little closer together and you can feel that some kind of relation exists between these two variables.

11.1.4 Scatter Diagram with Strong Correlation

This type of diagram is also known as “Scatter Diagram with High Degree of Correlation”. In this diagram, data points are grouped very close to each other such that you can draw a line by following their pattern.

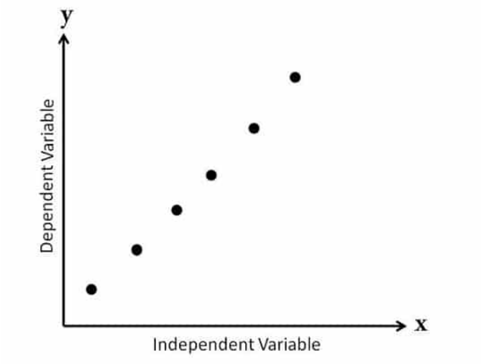


Figure 11.5: Scatter Diagram with Strong Correlation

In this case you will say that the variables are closely related to each other.Here the discussion on the first type of classification ends.

11.1.5 Scatter Diagram Base slope of trend of the Data Point

You can also divide the scatter diagram according to the slope of the trend of the data points, such as:

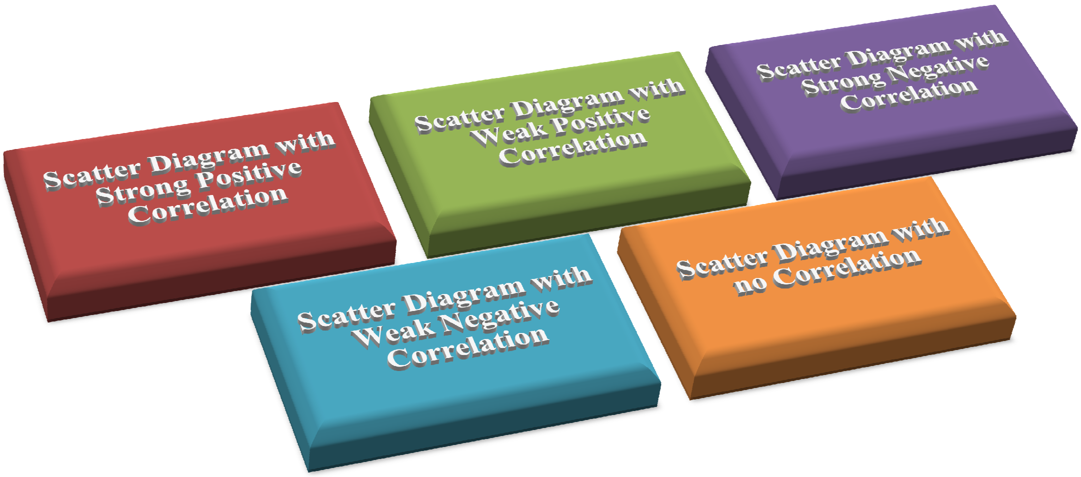

Scatter Diagram with Strong Positive Correlation
Scatter Diagram with Weak Positive Correlation
Scatter Diagram with Strong Negative Correlation
Scatter Diagram with Weak Negative Correlation
Scatter Diagram with no Correlation

Figure 11.6: Scatter Diagram Base slope of trend of the Data Point

Strong correlation means there is a clear visible relation and weak correlation means a visible relationship is not very clear.

1. **Scatter Diagram with Strong Positive Correlation**

This type of diagram is also known as Scatter Diagram with Positive Slant.

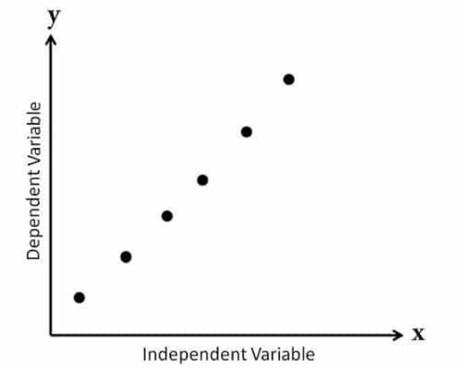


Figure 11.7: Scatter Diagram with Strong Positive Correlation

In positive slant, the correlation will be positive, i.e. as the value of x increases, the value of y will also increase. Hence you can say that the slope of straight line drawn along the data points will go up. The pattern will resemble the straight line. For example, if the temperature goes up, cold drink sales will also go up.

2. **Scatter Diagram with Weak Positive Correlation**

   Here as the value of x increases the value of y will also increase, but the pattern will not closely resemble a straight line.

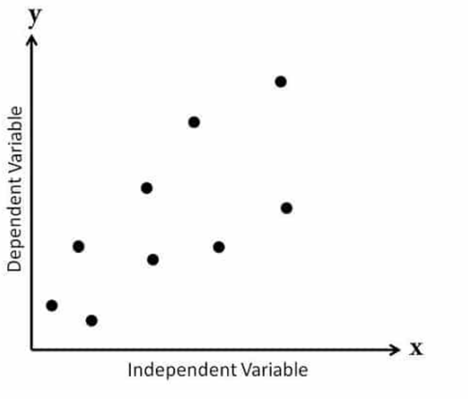


Figure 11.8: Scatter Diagram with weak Positive Correlation

3. **Scatter Diagram with Strong Negative Correlation**

   This type of diagram is also known as Scatter Diagram with Negative Slant.

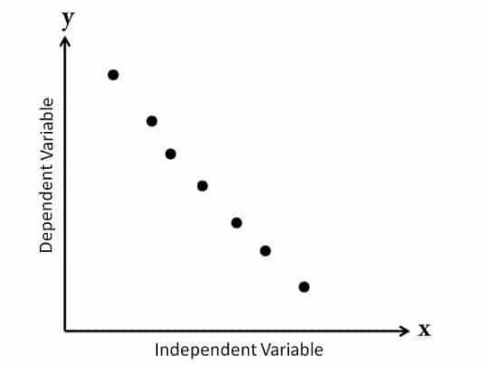


Figure 11.9: Scatter Diagram with Strong Positive Correlation

In negative slant, the correlation will be negative, i.e. as the value of x increases, the value of y will decrease. Here you can say that the slope of a straight line drawn along the data points will go down. For example, if the temperature goes up, sales of entry tickets to the goes down.

4. **Scatter Diagram with Weak Negative Correlation**

Here as the value of x increases the value of y will decrease, but the pattern will not resemble a straight line.

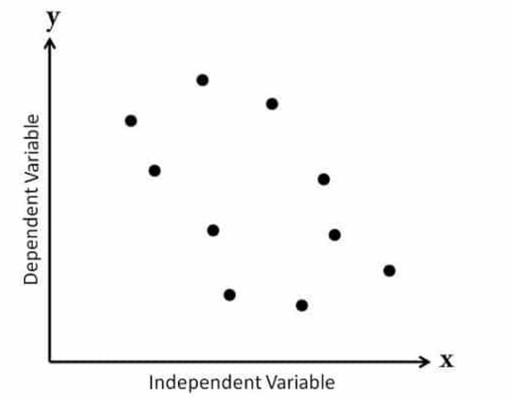


Figure 11.10: Scatter Diagram with Weak Negative Correlation

11.1.6 Benefits of a Scatter Diagram

1. The following are a few advantages of a scatter diagram:
2. It shows the relationship between two variables.
3. It is the best method to show you a non-linear pattern.
4. The range of data flow, i.e. maximum and minimum value, can be easily determined.

5. Observation and reading is straightforward.
6. Plotting the diagram is relatively simple.

11.1.7 Limitations of a Scatter Diagram

The following are a few limitations of a scatter diagram:

- ✓ Scatter diagram is unable to give you the exact extent of correlation.
- ✓ Scatter diagram does not show you the quantitative measure of the relationship between the variable. It only shows the quantitative expression of the quantitative change.
- ✓ This chart does not show you the relationship for more than two variables.

11.1.8 Simple Bivariate Regression Model

A regression model may be simple (linear), multiple or linear. We shall consider the simple Bivariate linear regression model, which is of the form:

$Y = \alpha + \beta x + e$ ……………………*

Where
Y is the dependent variable
X is the independent variable
α is the intercept
β is the slope of the line
e is the error term

The equation * is referred to as Least Square Equation. However, both α and β are constants and are determined by solving the simultaneous re-equations

$\sum y = \alpha n + \beta \sum x$ ……………………..(i)

$\sum xy = \alpha \sum x + \beta \sum x^2$ ………………(ii)

The two equations are the normal equations for the least square line. Solving the equations simultaneously, the values of α and β are obtained as: $\hat{\beta} = \frac{N\sum xy - \sum x \sum y}{N\sum x^2 - (\sum x)^2}$, $\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x}$

Computationally, the slope could be calculated as $\hat{\beta} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$

Example

Given the table below on the pair of random variables (x, y)

X	1	2	3	4	5
Y	2	3	4	6	8

Present the information on a scatter diagram and fit a least square line to the data

Solution

X	**Y**	**XY**	**X^2**	**Y^2**
1	2	2	1	4
2	3	6	4	9
3	4	12	9	16
4	6	24	18	36
5	8	40	25	64
15	**23**	**84**	**55**	**129**

$\sum XY = 84$

$\bar{X} = 15/5 = 3.0$

$\bar{Y} = 23/5 = 4.6$

To fit a line $y = \alpha + \beta x$

$$\beta = \frac{N\sum xy - \sum x \sum y}{N\sum x^2 - (\sum x)^2}$$ =(84 - (15.23)/5)/(55 – (15.5)/5)=1.5

$\alpha = \bar{y} - \beta\bar{x}$ 4.6 – (1.5)(3.0)= 0.1

Y=0.1+1.5x

The Least Square line can be fitted by substituting the values of x into $y = \alpha + \beta x$ to obtain corresponding y

11.2 Least Square Method

The least squares method is a form of mathematical regression analysis that finds the line of best fit for a dataset, providing a visual demonstration of the relationship between the data points. Each point of data is representative of the relationship between a known independent variable and an unknown dependent variable.

11.2.1 Breaking Down 'Least Squares Method'

The least squares method provides the overall rationale for the placement of the line of best fit among the data points being studied.

The most common application of the least squares method, referred to as linear or ordinary, aims to create a straight line that minimizes the sum of the squares of the errors generated by the results of the associated equations, such as the squared residuals resulting from differences in the observed value and the value anticipated based on the model.

This method of regression analysis begins with a set of data points to be graphed. An analyst using the least squares method will be seeking a line of best fit that explains the potential relationship between an independent variable and a dependent variable.

In regression analysis, dependent variables are designated on the vertical Y axis and independent variables are designated on the horizontal X axis. These designations will form the equation for the line of best fit, which is determined from the least squares method.

Example

The following are 8 data points that shows the relationship between the number of fishermen and the amount of fish (in thousand pounds) they can catch a day.

Number of Fishermen	Fish Caught
18	39
14	9
9	9
10	7
5	8
22	35
14	36
12	22

According to this data set, what is the function between the number of fishermen and the amount of fish caught?

Hint: let the number of fisherman be x, and the amount of fish caught be y, and use LLS to find the coefficients.

Answer

By the simple calculation and statistic knowledge, we can easily find out:

1. = 13
2. = 20.625, and
3. the following chart

X	Y				
18	39	5	18.375	91.875	25
14	9	1			1
9	9			46.5	16
10	7			40.875	9
5	8			101	64
22	35	9	14.375	129.375	81
14	36	1	15.375	15.375	1
12	22		1.375		1

Thus, we have , and , so the slope, a, = .

And last the intercept, b, = Therefore, the linear least-squares line is .The line of best fit determined from the least squares method has an equation that tells the story of the relationship between the data points.

Computer software models are used to determine the line of best fit equation, and these software models include a summary of outputs for analysis. The least squares method can be used for determining the line of best fit in any regression analysis. The coefficients and summary outputs explain the dependence of the variables being tested.

Summary

In this study session you have learnt about:

1.Scatter Diagram

Scatter graph method is a graphical technique of separating fixed and variable components of mixed cost by plotting activity level along x-axis and corresponding total cost (mixed cost) along y-axis. A regression line is then drawn on the graph by visual inspection.

2.Least Square Method

The least squares method is a form of mathematical regression analysis that finds the line of best fit for a dataset, providing a visual demonstration of the relationship between the data points. Each point of data is representative of the relationship between a known independent variable and an unknown dependent variable.

Self-Assessment Questions (SAQs) for study session 11

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 11.1 (Testing Learning Outcomes 11.1)

Highlight on the Scatter Diagram Method

SAQ 11.2 (Testing Learning Outcomes 11.2)

Discuss on the Least Square Method

Reference

John, E. F.(1974). Modern Elementary Statistics, International Edition. London: Prentice Hall.

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications

Shangodoyin D. K. et al (2002). Statistical Theory and Methods Ibadan: Joytal Press.

Study Session 12: Correlation Analysis

Introduction

Correlation is another way of assessing the relationship between variables. To be more precise, it measures the extent of correspondence between the ordering of two random variables. There is a large amount of resemblance between regression and correlation but for their methods of interpretation of the relationship. For example, a scatter diagram is of tremendous help when trying to describe the type of relationship existing between two variables.

This study session introduces you to the concept of correlation. The definition and types of correlation are highlighted. Interpretation of correlation coefficient is discussed. Examples are given for the two types of correlation discussed.

Learning Outcomes for Study Session 12

At the end of this study session, you should be able to:

12.1 Explain Pearson Product-Moment Correlation

12.2 Discuss Spear's Ranking Order Correlation

12.1 Pearson Product Correlation

The Pearson product-moment correlation coefficient (or Pearson correlation coefficient, for short) is a measure of the strength of a linear association between two variables and is denoted by *r*. Basically, a Pearson product-moment correlation attempts to draw a line of best fit through the data of two variables, and the Pearson correlation coefficient, *r*, indicates how far away all these data points are to this line of best fit (i.e., how well the data points fit this new model/line of best fit).

12.1.1 What values can the Pearson correlation coefficient take

The Pearson correlation coefficient, *r*, can take a range of values from +1 to -1. A value of 0 indicates that there is no association between the two variables. A value greater than 0 indicates a positive association; that is, as the value of one variable increases, so does the value of the other variable. A value less than 0 indicates a negative association; that is, as the value of one variable increases, the value of the other variable decreases. This is shown in the diagram below:

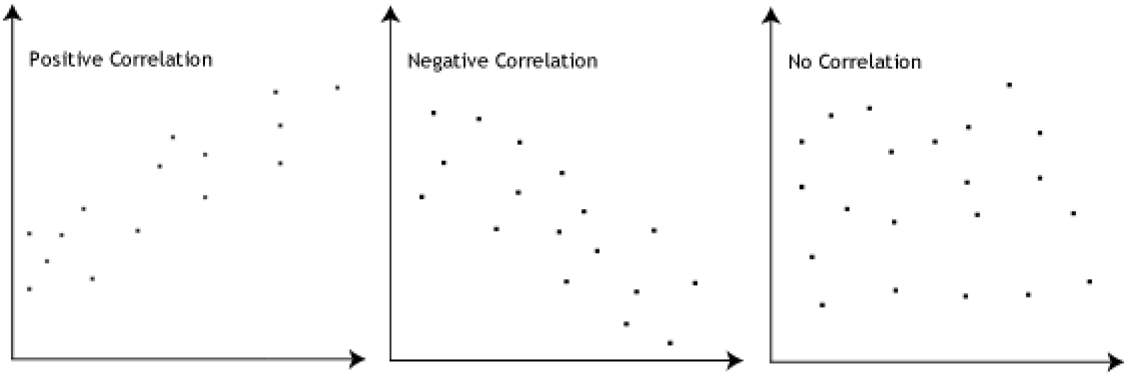


Figure 12.1: Types of Pearson Correlation

12.1.2 How can we determine the strength of association based on the Pearson correlation coefficient

The stronger the association of the two variables, the closer the Pearson correlation coefficient, *r*, will be to either +1 or -1 depending on whether the relationship is positive or negative, respectively. Achieving a value of +1 or -1 means that all your data points are included on the line of best fit – there are no data points that show any variation away from this line.

Values for *r* between +1 and -1 (for example, *r* = 0.8 or -0.4) indicate that there is variation around the line of best fit. The closer the value of *r* to 0 the greater the variation around the line of best fit. Different relationships and their correlation coefficients are shown in the diagram below:

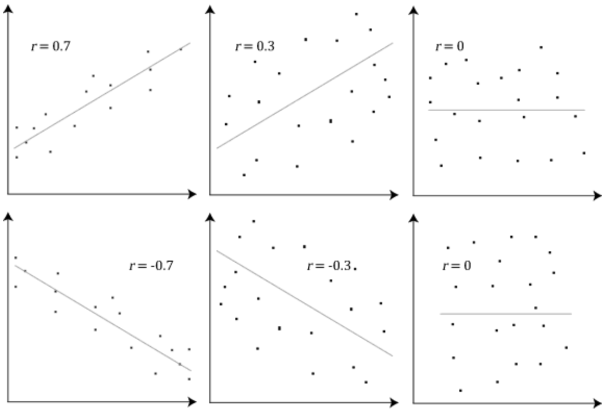


Figure 12.2: Pearson correlation coefficient

12.1.3 Guidelines to interpreting Pearson's correlation coefficient

Yes, the following guidelines have been proposed:

	Coefficient, *r*	
Strength of Association	Positive	Negative
Small	.1 to .3	-0.1 to -0.3
Medium	.3 to .5	-0.3 to -0.5
Large	.5 to 1.0	-0.5 to -1.0

Remember that these values are guidelines and whether an association is strong or not will also depend on what you are measuring.

12.1.4 Using any type of Variable for Pearson's Correlation Coefficient

No, the two variables have to be measured on either an interval or ratio scale. However, both variables do not need to be measured on the same scale (e.g., one variable can be ratio and one can be interval). Further information about types of variable can be found in our Types of Variable guide. If you have ordinal data, you will want to use Spearman's rank-order correlation or a Kendall's Tau Correlation instead of the Pearson product-moment correlation.

12.1.5 Do the two variables have to be measured in the same units

No, the two variables can be measured in entirely different units. For example, you could correlate a person's age with their blood sugar levels. Here, the units are completely different; age is measured in years and blood sugar level measured in mmol/L (a measure of concentration).

Indeed, the calculations for Pearson's correlation coefficient were designed such that the units of measurement do not affect the calculation. This allows the correlation coefficient to be comparable and not influenced by the units of the variables used.

12.1.6 What about dependent and independent variables

The Pearson product-moment correlation does not take into consideration whether a variable has been classified as a dependent or independent variable. It treats all variables equally. For example, you might want to find out whether basketball performance is correlated to a person's height. You might, therefore, plot a graph of performance against height and calculate the Pearson correlation coefficient.

Lets say, for example, that r = .67. That is, as height increases so does basketball performance. This makes sense. However, if we plotted the variables the other way around and wanted to determine whether a person's height was determined by their basketball performance (which makes no sense), we would still get r = .67.

This is because the Pearson correlation coefficient makes no account of any theory behind why you chose the two variables to compare. This is illustrated below:

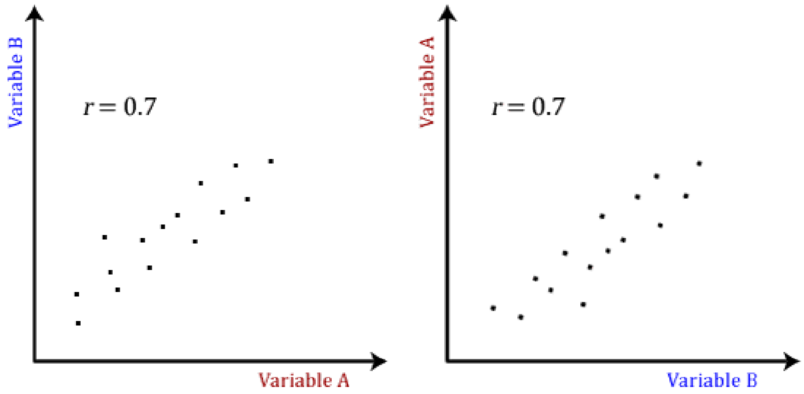


Figure 12.3: Comparison of Two Pearson Variables

12.1.7 The Pearson correlation coefficient indicate the slope of the line

It is important to realize that the Pearson correlation coefficient, r, does not represent the slope of the line of best fit. Therefore, if you get a Pearson correlation coefficient of +1 this does not mean that for every unit increase in one variable there is a unit increase in another. It simply means that there is no variation between the data points and the line of best fit. This is illustrated below:

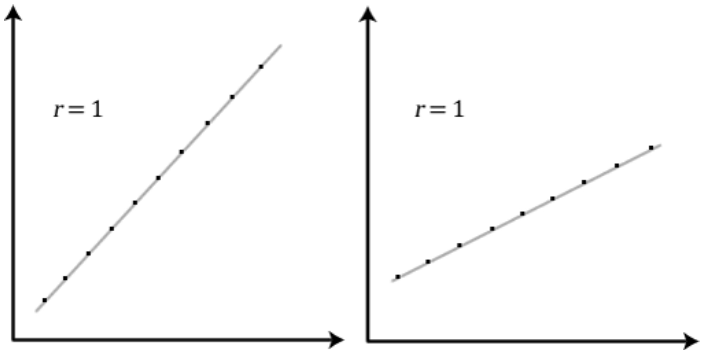


Figure 12.4: Pearson correlation coefficient indicate the slope of the line

12.1.8 Assumptions of Pearson's correlation make

There are five assumptions that are made with respect to Pearson's correlation:

1. The variables must be either interval or ratio measurements (see our Types of Variable guide for further details).
2. The variables must be approximately normally distributed (see our Testing for Normality guide for further details).
3. There is a linear relationship between the two variables (but see note at bottom of page). We discuss this later in this guide (jump to this section here).
4. Outliers are either kept to a minimum or are removed entirely. We also discuss this later in this guide (jump to this section here).
5. There is homoscedasticity of the data. This is discussed later in this guide (jump to this section here).

12.1.9 How can you detect a linear relationship

To test to see whether your two variables form a linear relationship you simply need to plot them on a graph (a scatte rplot, for example) and visually inspect the graph's shape. In the diagram below, you will find a few different examples of a linear relationship and some non-linear relationships. It is not appropriate to analyse a non-linear relationship using a Pearson product-moment correlation.

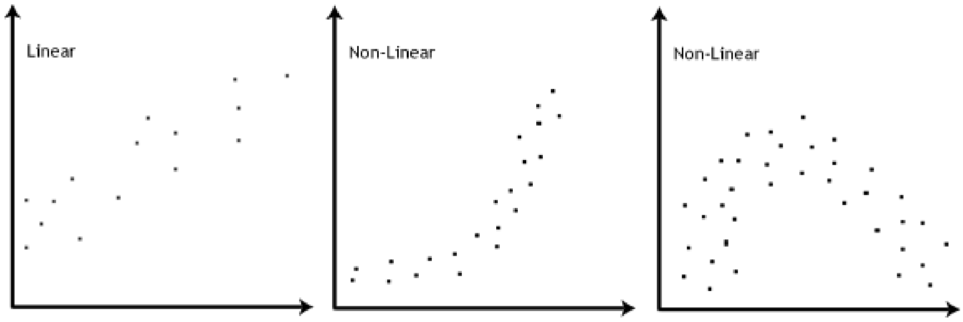


Figure 12.5: Pearson linear relationship

Note: Pearson's correlation determines the degree to which a relationship is linear. Put another way, it determines whether there is a linear component of association between two continuous variables. As such, linearity is not actually an assumption of Pearson's correlation.

However, you would not normally want to pursue a Pearson's correlation to determine the strength and direction of a linear relationship when you already know the relationship between your two variables is not linear.

Instead, the relationship between your two variables might be better described by another statistical measure. For this reason, it is not uncommon to view the relationship between your two variables in a scatter plot to see if running a Pearson's correlation is the best choice as a measure of association or whether another measure would be better.

In Text Question

To test to see whether two variables form a linear relationship you simply need to plot them on a table.**True/False**

In Text Answer

False (Graph)

12.2 Spear’s Ranking Order Correlation

In statistics, a rank correlation is any of several statistics that measure an ordinal association the relationship between rankings of different ordinal variables or different rankings of the same variable, where a "ranking" is the assignment of the labels "first", "second", "third", etc. to different observations of a particular variable.

A rank correlation coefficient measures the degree of similarity between two rankings, and can be used to assess the significance of the relation between them. This guide will tell you when you should use Spearman's rank-order correlation to analyse your data, what assumptions you have to satisfy, how to calculate it, and how to report it. If you want to know how to run a Spearman correlation in SPSS Statistics, go to our guide here.

12.2.1 When should you use the Spearman's rank-order correlation

The Spearman's rank-order correlation is the nonparametric version of the Pearson product-moment correlation. Spearman's correlation coefficient, (ρ, also signified by r_s) measures the strength and direction of association between two ranked variables.

12.2.2 What are the assumptions of the test

You need two variables that are either ordinal, interval or ratio (see our Types of Variable guide if you need clarification). Although you would normally hope to use a Pearson product-moment correlation on interval or ratio data, the Spearman correlation can be used when the assumptions of the Pearson correlation are markedly violated.

However, Spearman's correlation determines the strength and direction of the **monotonic relationship** between your two variables rather than the strength and direction of the linear relationship between your two variables, which is what Pearson's correlation determines.

12.2.3 What is a monotonic relationship

A monotonic relationship is a relationship that does one of the following: (1) as the value of one variable increases, so does the value of the other variable; or (2) as the value of one variable increases, the other variable value decreases. Examples of monotonic and non-monotonic relationships are presented in the diagram below:

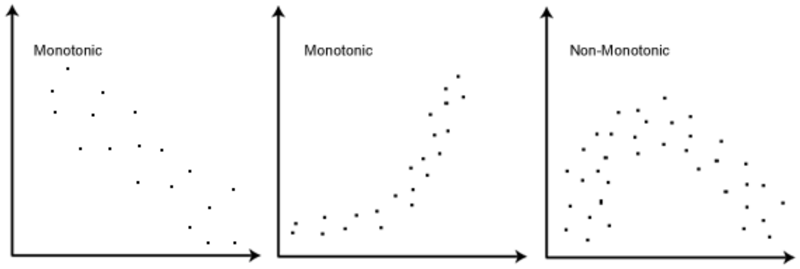


Figure 12.6: Spear’s Ranking Order Correlation monotonic relationship

12.2.4 Why is a monotonic relationship important to Spearman's correlation

Spearman's correlation measures the strength and direction of monotonic association between two variables. Monotonicity is "less restrictive" than that of a linear relationship. For example, the middle image above shows a relationship that is monotonic, but not linear.

A monotonic relationship is not strictly an assumption of Spearman's correlation. That is, you can run a Spearman's correlation on a non-monotonic relationship to determine if there is a **monotonic component** to the association.

However, you would normally pick a measure of association, such as Spearman's correlation, that fits the pattern of the observed data. That is, if a scatterplot shows that the relationship between your two variables looks monotonic you would run a Spearman's correlation because this will then measure the strength and direction of this monotonic relationship.

On the other hand if, for example, the relationship appears linear you would run a Pearson's correlation because this will measure the strength and direction of any linear relationship. You will not always be able to visually check whether you have a monotonic relationship, so in this case, you might run a Spearman's correlation anyway.

12.2.5 How to rank data

In some cases your data might already be ranked, but often you will find that you need to rank the data yourself. Thankfully, ranking data is not a difficult task and is easily achieved by working through your data in a table. Let us consider the following example data regarding the marks achieved in a maths and English exam:

Exam					**Marks**					
English	56	75	45	71	61	64	58	80	76	61
Maths	66	70	40	60	65	56	59	77	67	63

The procedure for ranking these scores is as follows:

First, create a table with four columns and label them as below:

English (mark)	**Maths (mark)**	**Rank (English)**	**Rank (maths)**
56	66	9	4
75	70	3	2
45	40	10	10

English (mark)	Maths (mark)	Rank (English)	Rank (maths)
71	60	4	7
61	65	6.5	5
64	56	5	9
58	59	8	8
80	77	1	1
76	67	2	3
61	63	6.5	6

You need to rank the scores for maths and English separately. The score with the highest value should be labelled "1" and the lowest score should be labelled "10" (if your data set has more than 10 cases then the lowest score will be how many cases you have). Look carefully at the two individuals that scored 61 in the English exam (highlighted in bold).

Notice their joint rank of 6.5. This is because when you have two identical values in the data (called a "tie"), you need to take the average of the ranks that they would have otherwise occupied. We do this because, in this example, we have no way of knowing which score should be put in rank 6 and which score should be ranked 7. Therefore, you will notice that the ranks of 6 and 7 do not exist for English. These two ranks have been averaged ((6 + 7)/2 = 6.5) and assigned to each of these "tied" scores.

12.2.6 What is the definition of Spearman's rank-order correlation?

There are two methods to calculate Spearman's correlation depending on whether: (1) your data does not have tied ranks or (2) your data has tied ranks. The formula for when there are no tied ranks is:

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

where d_i = difference in paired ranks and n = number of cases. The formula to use when there are tied ranks is:

$$\rho = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}}$$

where *i* = paired score.

It is denoted by

$$r = \frac{\sum XY - n\overline{X}\overline{Y}}{\sqrt{(\sum X^2 - n\overline{X}^2)(\sum Y^2 - n\overline{Y}^2)}}$$

= -1 ≤ r ≤ 1

Example: The demand and price of a particular commodity are recorded as follows:

Demand (Y)	2	3	5	4	6
Price N(X)	1	2	3	4	5

Find the product moment correlation co-efficient of the data.

Solution

Y_i	X_i	$X_i Y_i$	X_i^2	Y^2
2	1	2	1	4
3	2	6	4	9
5	3	15	9	25
4	4	16	16	16
6	5	30	25	36
20	15	96	55	90

$$r = \frac{\sum XY - n\overline{X}\overline{Y}}{\sqrt{(\sum X^2 - n\overline{X}^2)(\sum Y^2 - n\overline{Y}^2)}}$$

$$= \frac{90 - 5(3)(4)}{\sqrt{[(55 - 5(3^3))][(90 - 5(4^2)]}}$$

$$r = \frac{9}{\sqrt{10x10}} = \frac{9}{10} = 0.9$$

The above result r = 0.9 implies a strong (or direct) relationship (Correlation between) the price and demand for the commodity.

Interpretation of r

i. When r = 1, it indicates a perfect positive correlation.

ii. When r = -1, it indicates a perfect negative correlation.

iii. When –1 < r < -0.5, it indicates a strong negative correlation.

iv. When –0.5 < r < 0, it indicates a weak negative correlation.

v. When 0 < r < 0.5, it indicates a weak positive correlation.

vi. When 0.5 < r 1, it indicates a strong positive correlation.

Example

Suppose we are interested in seeing if there is a linear relationship between age and height for children ages 3 to 9.

A random sample of children gave the following data

Age (*x*)	Height (*y*)	*xy*
3	38	114
3	31	93
4	37	148
6	40	240
6	49	294
7	45	315
9	51	459
38	291	1663

So

$$\sum x = 38, \sum y = 291, \sum xy = 1663$$

Furthermore,

$$\sum x^2 = 236 \quad ,\text{and} \sum y^2 = 12401$$

Response variable (*y*) - the one we are primarily interested in.

Explanatory variable (*x*) - one that is thought to affect the response variable.

First, construct a scatter diagram (scatterplot) - a plot of the ordered pairs (*x*, *y*) with the response variable, *y*, on the vertical axis.

<u>Correlation coefficient</u> - measures the <u>strength</u> and <u>direction</u> of the <u>linear</u> relationship between two quantitative variables.

Notation: □ = population correlation

r = sample correlation

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2 - \left(\sum x\right)^2}\sqrt{n\sum y^2 - \left(\sum y\right)^2}}$$

So, for our data set,

$$r = \frac{7(1663) - (38)(291)}{\sqrt{7(236) - (38)^2}\sqrt{7(12401) - (291)^2}} = 0.877$$

Interpretation: There is a strong, positive linear relationship between age and height for children ages 3 to 9.

Coefficient of determination = r^2 = the proportion of variation in *y* that is explained by its linear relationship with *x*.

1

Interpretation: 76.9% of the variation in height can be explained by its linear relationship with age.

Summary

In this study session you have learnt about:

1. Pearson Product Correlation

The Pearson product-moment correlation coefficient (or Pearson correlation coefficient, for short) is a measure of the strength of a linear association between two variables and is denoted by *r*.

Basically, a Pearson product-moment correlation attempts to draw a line of best fit through the data of two variables, and the Pearson correlation coefficient, *r*, indicates how far away all these data points are to this line of best fit (i.e., how well the data points fit this new model/line of best fit).

1. Spear’s Ranking Order Correlation

In statistics, a rank correlation is any of several statistics that measure an ordinal association the relationship between rankings of different ordinal variables or different rankings of the same variable, where a "ranking" is the assignment of the labels "first", "second", "third", etc. to different observations of a particular variable.

Self-Assessment Questions (SAQs) for study session 12

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 12.1 (Testing Learning Outcomes 12.1)

Highlight on Pearson Product Correlation

SAQ 12.2 (Testing Learning Outcomes 12.2)

Discuss Spear‘s Ranking order Correlation

Reference

John, E. F.(1974). Modern Elementary Statistics, International Editions London: Prentice Hall

Murray, R. S. (1972) Schaum’s Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company.

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications.

Study Session 13: Introduction to Time Series Analysis

Introduction

A time series plays a significant role in the analysis of socio-economic data. These are mostly data on finance, insurance, marketing, population, etc. It is also very useful in forecasting because the objective of a time series is to forecast future values to enable us make good plans for the future.

A time series is a series of data points indexed (or listed or graphed) in time order. Most commonly, a time series is a sequence taken at successive equally spaced points in time. ... Time series forecasting is the use of a model to predict future values based on previously observed values.

The concept of Time Series is covered in this study session. It introduces you to the definition of Time Series, objectives of a Time Series, component of a Time Series and models of Time Series.

Learning Outcomes for Study Session 13

At the end of this study session, you should be able to:

13.1 Define Time Series

13.2 Component of Time Series

13.1 Definition of a Time Series

A time series consists of a set of chronological observation (continuous or discrete) taken at a specified time, usually at equal intervals. We denote a time series by (Xt) where (Xt) is the observed value in time.

A time series may be represented graphically as a plot, that is, the plot of observation Xt against time (t) so as to observe the inherent characteristics of the time series data. The basic idea of time series is that given a set of data, one should be able to estimate or forecast Xt+1, Xt+2, etc.

13.1.1 Objectives of Time Series Analysis

It would be incomplete to approach the fundamentals of time series analysis without discussing the main reasons for analyzing time series. The objectives of time series analysis may be broadly classified into four major types of investigation.

The first exploration of time series data is the plot of the series over time (Time Plot), then a sample descriptive measures of the main properties of the series could be obtained. In this, we intend to look at outliers, troughs, presence of turning points etc that may be pronounced on the time plot.

1. In order to have a deeper understanding of the mechanism, which generates a given time series, we make use of the variation in one time series to explain the variation in another time series. The analysis of linear systems provides more information.
2. An important task in time series analysis is to predict the future. Given an observed time series, one may want to predict the future values of the series. If one can forecast that a manufacturing process is going to move off target, then an appropriate corrective action could be taken.
3. When a time series generated measures the quality of a manufacturing process, the aim of the analysis may be to control the process. The procedures on these are of several kinds, ranging from charts to inspection models. A stochastic model is fitted to the series, and the future values of the series are provided and then the input process variables are adjusted so as to keep the process on target.

13.2 Components of Time Series

The factors that are responsible to bring about changes in a time series, also called the components of time series, are as follows:

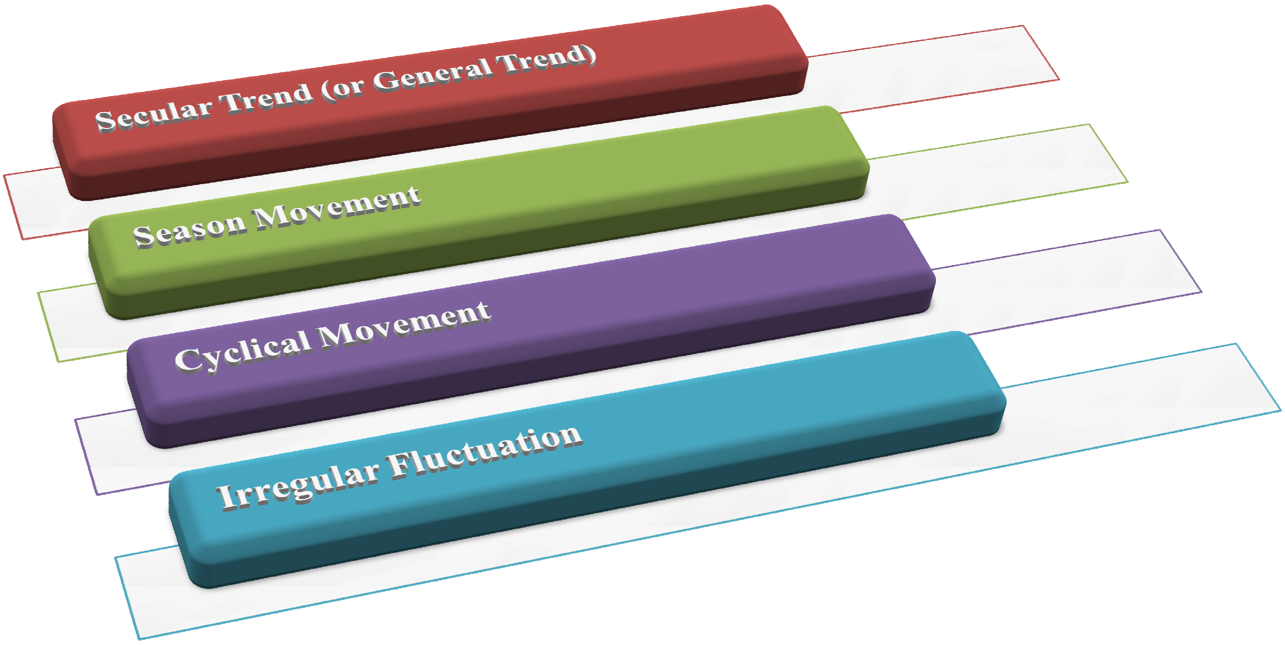


Figure 13.1: Component of Time Series

1.Secular Trend

The secular trend is the main component of a time series which results from long term effect of socio-economic and political factors. This trend may show the growth or decline in a time series over a long period. This is the type of tendency which continues to persist for a very long period. Prices, export and imports data, for example, reflect obviously increasing tendencies over time.

2.Seasonal Trend

These are short term movements occurring in a data due to seasonal factors. The short term is generally considered as a period in which changes occur in a time series with variations in weather or festivities. For example, it is commonly observed that the consumption of ice-cream during summer us generally high and hence sales of an ice-cream dealer would be higher in some months of the year while relatively lower during winter months.

Employment, output, export etc. are subjected to change due to variation in weather. Similarly sales of garments, umbrella, greeting cards and fire-work are subjected to large variation during festivals like Valentine’s Day, Eid, Christmas, New Year etc. These types of variation in a time series are isolated only when the series is provided biannually, quarterly or monthly.

3.Cyclic Movements

These are long term oscillation occurring in a time series. These oscillations are mostly observed in economics data and the periods of such oscillations are generally extended from five to twelve years or more. These oscillations are associated to the well known business cycles. These cyclic movements can be studied provided a long series of measurements, free from irregular fluctuations is available.

4.Irregular Fluctuations

These are sudden changes occurring in a time series which are unlikely to be repeated, it is that component of a time series which cannot be explained by trend, seasonal or cyclic movements .It is because of this fact these variations some-times called residual or random component.

These variations though accidental in nature, can cause a continual change in the trend, seasonal and cyclical oscillations during the forthcoming period. Floods, fires, earthquakes, revolutions, epidemics and strikes etc,. are the root cause of such irregularities.

In Text Question

The short term is generally considered as a period in which changes occur in a time series with variations in weather or festivities.

(a) Irregular

(b) Cyclic Movement

(c) Seasonal Trend

(d) Secular Trend

In Text Answer

The answer is (c) seasonal Trend

13.2.1 Time series decomposition

We shall think of the time series $y_t$yt as comprising three components: a seasonal component, a trend-cycle component (containing both trend and cycle), and a remainder component (containing anything else in the time series). For example, if we assume an additive model, then we can write

$y_t=S_t+T_t+E_t$,yt=St+Tt+Et,

where $y_t$yt is the data at period tt, S_tSt is the seasonal component at period tt, T_tTt is the trend-cycle component at period tt and E_tEt is the remainder (or irregular or error) component at period tt. Alternatively, a multiplicative model would be written as

$y_t=S_t×T_t×E_t$.yt=St×Tt×Et.

The additive model is most appropriate if the magnitude of the seasonal fluctuations or the variation around the trend-cycle does not vary with the level of the time series. When the variation in the seasonal pattern, or the variation around the trend-cycle, appears to be proportional to the level of the time series, then a multiplicative model is more appropriate. With economic time series, multiplicative models are common.

An alternative to using a multiplicative model, is to first transform the data until the variation in the series appears to be stable over time, and then use an additive model.

Sometimes, the trend-cycle component is simply called the “trend” component, even though it may contain cyclic behaviour as well.

The data have been adjusted by working days and normalized so a value of 100 corresponds to 2005.

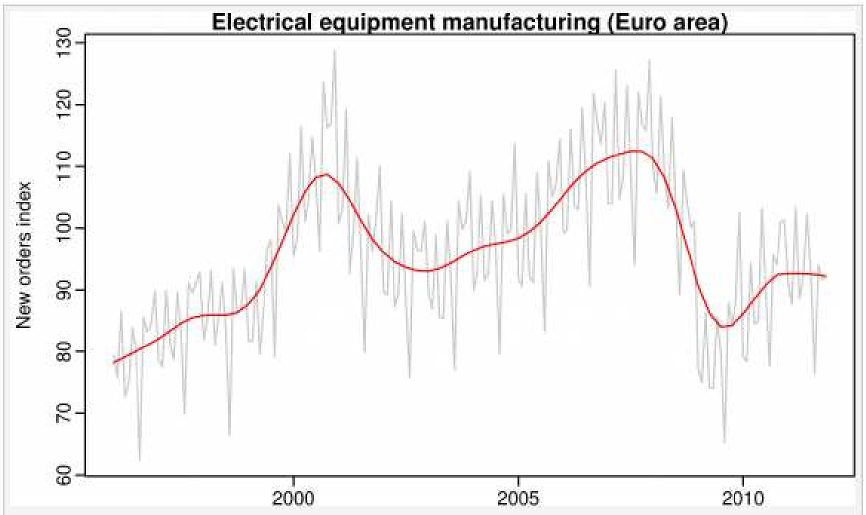


Figure 13.2: Time series Decomposition 1

13.2.2 De-Seasonalizing a Time Series

Seasonality in a time series can be identified by regularly spaced peaks and troughs which have a consistent direction and approximately the same magnitude every year, relative to the trend. The graph below, that of a retailer, shows a strongly seasonal series. In the fourth quarter each year, sales increase due to holiday shopping. In this example, the magnitude of the seasonal component increases over time, as does the trend.

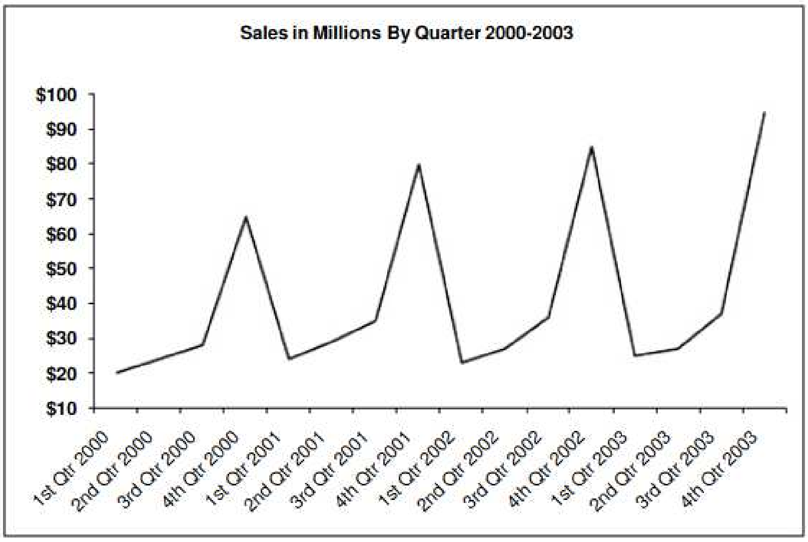


Figure 13.3: Time series Decomposition 2

A time series can be de-seasonalized when only a seasonal component is present, or when both seasonal and trend components are present. This is a two-step process:

1.Compute seasonal/irregular indexes and use them to de-seasonalize the data;

2.Use regression analysis on the remaining trend data if a trend is apparent in

13.2.3 Seasonal Variation

In statistics, many time series exhibit cyclic variation known as seasonality, seasonal variation, periodic variation, or periodic fluctuations. This variation can be either regular or semi-regular. Seasonal variation is a component of a time series which is defined as the repetitive and predictable movement around the trend line in one year or less.

It is detected by measuring the quantity of interest for small time intervals, such as days, weeks, months or quarters. Organizations facing seasonal variations, like the motor vehicle industry, are often interested in knowing their performance relative to the normal seasonal variation.

The same applies to the ministry of employment which expects unemployment to increase in June because recent graduates are just arriving into the job market and schools have also been given a vacation for the summer. That unemployment increased as predicted is a moot point; the relevant factor is whether the increase is more or less than expected.

13.2.4 Moving Average

A widely used indicator in technical analysis that helps smooth out price action by filtering out the “noise” from random price fluctuations. A moving average (MA) is a trend-following or lagging indicator because it is based on past prices.

The two basic and commonly used MAs are the simple moving average (SMA), which is the simple average of a security over a defined number of time periods, and the exponential moving average (EMA), which gives bigger weight to more recent prices.

The most common applications of MAs are to identify the trend direction and to determine support and resistance levels. While MAs are useful enough on their own, they also form the basis for other indicators such as the Moving Average Convergence Divergence (MACD).

13.2.5 Additive model

A data model in which the effects of individual factors are differentiated and added together to model the data. They occur in several Minitab commands:

- An additive model is optional for Decomposition procedures and for Winters' method.
- An additive model is optional for two-way ANOVA procedures. Choose this option to omit the interaction term from the model.

13.2.6 Multiplicative model

This model assumes that as the data increase, so does the seasonal pattern. Most time series plots exhibit such a pattern. In this model, the trend and seasonal components are multiplied and then added to the error component.

13.2.7 Using an additive model or a multiplicative model

Choose the multiplicative model when the magnitude of the seasonal pattern in the data depends on the magnitude of the data. In other words, the magnitude of the seasonal pattern increases as the data values increase, and decreases as the data values decrease.

Choose the additive model when the magnitude of the seasonal pattern in the data does not depend on the magnitude of the data. In other words, the magnitude of the seasonal pattern does not change as the series goes up or down.

If the pattern in the data is not very obvious, and you have trouble choosing between the additive and multiplicative procedures, you can try both and choose the one with smaller accuracy measures.

Time Series Models

In the light of the above components of a time series $\{X_t\}$, we can represent a series in two convenient models as some functions of the component. The standard forms are the following:

1. $X_t = T_t + C_t + S_t + I_t$. called the additive model an
2. $X_t = T_t . C_t . S_t . I_t$. called the multiplicative model.

The logarithmic transformation of (1) leads to (2), and where the cyclic variation is not pronounced, the two models reduce to $X_t = T_t . C_t . I_t$. and $T_t + C_t + I_t$ respectively.

Summary

In this study session you have learnt about:

1. Definition of a Time Series

A time series consists of a set of chronological observation (continuous or discrete) taken at a specified time, usually at equal intervals. We denote a time series by (Xt) where (Xt) is the observed value in time. A time series may be represented graphically as a plot, that is, the plot of observation Xt against time (t) so as to observe the inherent characteristics of the time series data.

2.Component of Time Series

- Secular Trend
- Seasonal Trend

- Cyclic Movements
- Irregular Fluctuations

Self-Assessment Questions (SAQs) for study session 13

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 13.1 (Testing Learning Outcomes 13.1)

Explain Time Series

SAQ 13.2 (Testing Learning Outcomes 13.2)

Discuss the Component of Time Series

References

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications.

Shangodoyin D. K. and Ojo, J. F. (200). Elements of Time Series Analysis Ibadan: Rasmed Publications.

Shangodoyin D. K. et al (2002). Statistical Theory and Methods Ibadan: Joytal Printing Press.

Study Session 14: Time Series Analysis - Estimation of Trends and Seasonal Variations

Introduction

Trend estimation is a statistical technique to aid interpretation of data. When a series of measurements of a process are treated as a time series, trend estimation can be used to make and justify statements about tendencies in the data, by relating the measurements to the times at which they occurred.

The objective of time series analysis is to identify the direction and magnitude of any trend present in the data. To achieve this objective, many methods are available in practice for the estimation of trend in a series. We shall examine some of these methods

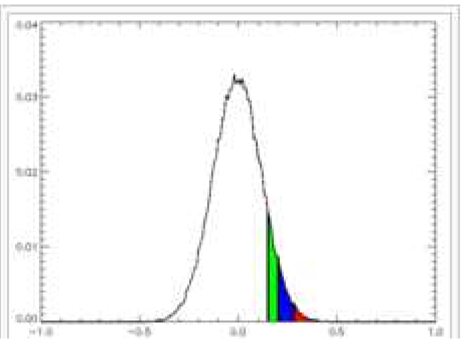


Figure 14.1: Estimation Trend

This study session will introduce you to estimation of trends and seasonal variation in time series analysis.

Learning Outcomes for Study Session 14

At the end of this study session, you should be able to:

14.1 Highlight on Moving Average Method

14.2 Explain The Least Square Method

14.1 Moving Average Method

The technique of using the moving average method is to replace a particular measurement by the arithmetic mean of a series of measurements of which it is the center. When an odd number is chosen, the moving average is centred as an observed measurement.

But where an even number of measurement is chosen, the moving average is centred between two observed measurements and must be re-centred before comparisons can be made between the average and the measurement.

Suppose we are given $X_1, X_2, \ldots, X_N$ on $\{X_t\}$, the n-point moving average are:

$$Y_1 = \frac{X_{1+} X_{2+} \ldots + X_N}{n}$$

$$Y_2 = \frac{X_{1+} X_{2+} \ldots + X_N}{n}$$

$$. \qquad\qquad .$$

$$Y_{N-n} = \frac{X_{N-n} + X_{N-n} + \ldots + X_N}{n}$$

For illustration, consider the hypothetical observations $X_1, X_2, \ldots, X_{12}$ given on $\{X_t\}$. Suppose our interest is on 4pt-moving average, we shall make use of the routine displayed in the table below:

DATA	4-POINT TOTAL	4-POINT AVERAGE	CENTRED 4-POINT MOVING AVERAGE
X_1	4		

	$\sum_{i=1} Xi = T_1$		
X_2	$\sum_{i=2}^{5} Xi = T_2$	$T_1/4 = V_1$	
X_3	$\sum_{i=3}^{6} Xi = T_3$	$T_2/4 = V_2$	$V_1 + V_2 = M_1$
X_4		$T_3/4 = V_3$	$V_2 + V_3 = M_2$
X_5	$\sum_{i=4}^{7} Xi = T_4$	$T_4/4 = V_4$	$V_3 + V_4 = M_3$
X_6	$\sum_{i=5}^{8} Xi = T_5$	$T_5/4 = V_5$	$V_4 + V_5 = M_4$
X_7	$\sum_{i=6}^{9} Xi = T_6$	$T_6/4 = V_6$	$V_5 + V_6 = M_5$
X_8	$\sum_{i=7}^{10} Xi = T_7$	$T_7/4 = V_7$	$V_6 + V_7 = M_6$
X_9	$\sum_{i=8}^{11} Xi = T_8$	$T_8/4 = V_8$	$V_7 + V_8 = M_7$
X_{10}	$\sum_{i=9}^{12} Xi = T_9$	$T_9/4 = V_9$	$V_8 + V_9 = M_8$
X_{11}			
X_{12}			

The advantage is that it gives the true nature of the trend, that is, a simple description of the underlying trend, particularly, when the trend is small. Although, this method loses the extreme values, it should be noted that one can generalize the moving average trend for the series to any point in time.

14.2 Method of Least Squares

This method can be used to find the equation of an appropriate trend line and the trend values Tt can be computed from the equation using the least squares approach of regression analysis. The normal equations for the trend line are:

$$\sum_{t=1}^{n} X_t = na + b\sum_{t=1}^{n} t$$

$$\sum_{t=1}^{n} tX_t = a\sum_{t=1}^{n} t + b\sum_{t=1}^{n} t^2$$

The least squares estimate of a and b are the solution to the normal equations which can be determined by solving the equations simultaneously. The estimates are then given as

$$\hat{a} = \overline{X}_t - b\bar{t}$$

$$\bar{b} = \frac{\sum tX_t - \frac{(\sum t)(\sum X_t)}{n}}{\sum t^2 - \frac{(\sum t)^2}{n}}$$

Thus the trend line equation is then

$$T_t = \hat{a} + \hat{b}t$$

One disadvantage of this method is that a linear trend may be fitted into a series which is not exactly linear.

14.2.1 The Freehand Method

This consists of fitting a trend line or curve by looking at the graph. If the type of equation of this curve is known, it is possible to obtain the constants in the equation by choosing as many points on the curve in a straight line; two points are necessary. The disadvantage of this method is that different observers will obtain different curves and equations.

14.2.2 The Method of Semi Average

This consists of separating the data into two equal parts and averaging the data in each part, thus obtaining two points on the graph of the time series. A trend line is then drawn between these two points, and trend values can be determined. The demerit of this method is that it may lead to poor results when used indiscriminately. Also, it is applicable only where the trend is linear.

14.2.3 The Method of Curve Fitting

This approach is particularly adequate when the series is non-seasonal and has trend. The various curves that can be fitted are Gompertz, Logistic and Polynomial.

1.Elimination of Trend

A special way of eliminating trend is through filtering (which is particularly useful for removing a trend). This is done to make a series stationary, the approach is by differencing the data based on the nature of the curve.For non-seasonal data, first-order differencing is sufficient to attain stationary, so that a new series is generated. Suppose we have a linear curve given as:

$$x_t = a + bt$$

To remove trend we make use of the first difference defined as:

$$\begin{aligned} Y_t &= \Delta x_t = x_t - x_{t-1} \\ &= a + bt - (a + bt - b) \\ &= b. \quad \text{Constant and independent of time.} \end{aligned}$$

Thus in general to eliminate the nth order polynomial trend we use

$$^{n}\, x_t$$

14.2.4 Estimation of Seasonal Variation

A. Multiplicative Model

Suppose we assume a multiplicative model of the form

$$X_t = T_t \,.\, S_t \,.\, C_t \,.\, I_t \text{ or}$$

$$X_t = T_t(C_t) \,.\, S_t \,.\, I_t$$

Having estimated the trend, the seasonal variation at time t is

$$S_t = X_t / T_t$$

B. Additive Model

Suppose a series X_t has an additive relationship with the time series components, that is,

$$X_t = T_t + S_t + C_t + I_t \text{ or}$$

$$X_t = T_t(C_t) + S_t + I_t$$

Then the seasonal variation at time t is

$$S_t = X_t - T_t$$

Example 1

The time series shown below shows the cost of production of Associated Matches Industry for a four-year period, spanning 1988 - 1992.

cost of production (_'000s) of ami

Year	**I**	**II**	**III**	**IV**
1988			55.8	53.6
1989	55.5	57.5	57.2	54.6
1990	56.3	56	55.4	50.4
1991	56.1	57.5	53.9	53.4
1992	60.1	63		

By using the method of moving averages, calculate the trend of the production cost. Plot both sets of figures on a graph.

Solution

Year	**Quarter**	**(i)** **Costs**	**(ii)** **4-quarter Totals**	**(iii)** **4-quarter total in pairs**	**(iv)** **Trend (iii□8)**
1988	III	55.8			
	IV	53.6	222.4		
1989	I	55.5	223.8	446.2	55.8
	II	57.5	224.8	448.6	56.1
	III	57.2	225.6	450.4	56.3
	IV	54.6	224.1	449.7	56.2
1990	I	56.3	222.3	446.4	55.8
	II	56	218.1	440.4	55.1

	III	55.4	217.9	436	54.5
	IV	50.4	219.4	437.3	54.7
1991	I	56.1	217.9	437.3	54.7
	II	57.5	220.9	438.8	54.9
	III	53.9	224.9	445.8	55.7
	IV	53.4	230.4	455.3	56.9
1992	I	60.1			
	II	63			

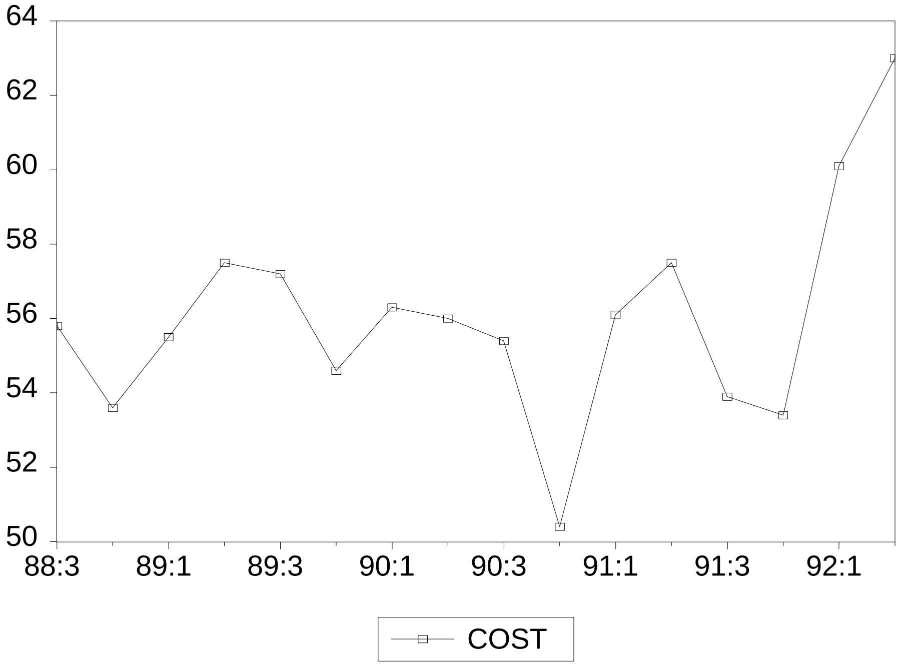


Fig. 1-1: Graph of cost of production of AMI (1988 - 1992) and its corresponding trend values using moving averages method

Summary

In this study session you have learnt about:

1. **Moving Average Method**

   The technique of using the moving average method is to replace a particular measurement by the arithmetic mean of a series of measurements of which it is the center. When an odd number is chosen, the moving average is centred as an observed measurement.

   But where an even number of measurement is chosen, the moving average is centred between two observed measurements and must be re-centred before comparisons can be made between the average and the measurement.

2. **Method of Least Squares**

   This method can be used to find the equation of an appropriate trend line and the trend values Tt can be computed from the equation using the least squares approach of regression analysis. The normal equations for the trend line are:

$$\sum_{t=1}^{n} X_t = na + b\sum_{t=1}^{n} t$$

$$\sum_{t=1}^{n} tX_t = a\sum_{t=1}^{n} t + b\sum_{t=1}^{n} t^2$$

The least squares estimate of a and b are the solution to the normal equations which can be determined by solving the equations simultaneously. The estimates are then given as

$$\hat{a} = \overline{X}_t - b\bar{t}$$

$$\bar{b} = \frac{\sum tX_t - \frac{(\sum t)(\sum X_t)}{n}}{\sum t^2 - \frac{(\sum t)^2}{n}}$$

Self-Assessment Questions (SAQs) for study session 14

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 14.1 (Testing Learning Outcomes 14.1)

Explain moving average Method

SAQ 14.2 (Testing Learning Outcomes 14.2)

Discuss Least Square Method

References

John, E. F.(1974). Modern Elementary Statistics. International Edition. London: Prentice Hall

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications.

Shangodoyin D. K. and Ojo, J. F. (200). Elements of Time Series Analysis Ibadan: Rasmed Publications.

Study Session 15: Time Series Analysis - Seasonal Indices and Forecasting

Introduction

A seasonal index is a measure of how a particular season compares with the average season. Seasonal indices are calculated so that their average is 1. This means that the sum of the seasonal indices equals the number of seasons. To calculate de-seasonalised data, each entry is divided by its seasonal index as follows.

Learning Outcomes From Study Session 15

15.1 Define Forecasting

15.2 Explain Elementary Forecasting

15.1 Define Forecasting

A planning tool that helps management in its attempts to cope with the uncertainty of the future, relying mainly on data from the past and present and analysis of trends. Forecasting starts with certain assumptions based on the management experience, knowledge, and judgment.

These estimates are projected in the coming months or years using one or more techniques such as Box-Jenkins models, Delphi method, exponential smoothing, moving averages, regression analysis, and trend projection. Since any error in the assumptions will result in a similar or magnified error in forecasting, the technique of sensitivity analysis is used which assigns a range of values to the uncertain factors (variables).

15.1.1 Seasonal Indices

This is a set of numbers that shows the relative values of a variable during the months of the year or quarters of a year. A suitable method is the Average Percentage Method; this method involves the expression of the data as percentages of the total for the period. The percentages for the corresponding months (quarters) of different years are then averaged, using either a **mean** or **median** (If the mean is used, it helps to avoid extreme values, which may occur).

The resulting 12(4) percentages give the seasonal index. Depending on the model used, the resulting total should be 1200% or 0% for multiplicative model, and 400% for additive model. But if this is not so, a suitable factor should be used to adjust.

Consider the hypothetical figure (X_{ij}) on quarterly sales in a company for four years:

Quarter

YEAR	1	2	3	4
1997	X_{11} (S_1)	X_{12} (S_2)	X_{13} (S_3)	X_{14} (S_4)
1998	X_{21} (S_5)	X_{22} (S_6)	X_{23} (S_7)	X_{24} (S_8)
1999	X_{31} (S_9)	X_{32} (S_{10})	X_{33} (S_{11})	X_{34} (S_{12})
2000	X_{41} (S_{13})	X_{42} (S_{14})	X_{43} (S_{15})	X_{44} (S_{16})
TOTAL	$S_1+S_5+S_9+S_{13}= ST_1$	$S_2+S_6+S_{10}+S_{14} =ST_2$	$S_3+S_7+S_{11}+S_{15} = ST_3$	$S_4+S_8+S_{12}+S_{16} = ST_4$
Average	$\frac{ST_1}{4}$ x 100 = SI_1	$\frac{ST_2}{4}$ x 100 = SI_2	$\frac{ST_3}{4}$ x 100 = SI_3	$\frac{ST_4}{4}$ x 100 = SI_4

15.2 Elementary Forecasting

Very often, we have a series of measurements, which are affected by the same time series components, such as trend and seasonal variation; these are accounted for in forecasting. Any change in the more sensitive series will anticipate the corresponding change in the components therein, and can be used as a forecasting indicator.

Generally, the forecasting for period p is

$X_p = T_p$ x SI_p and

$X_p = T_p + SI_p$ for both the multiplication and additive models respectively.

Example 1
The following information has been supplied by the sales department

		1990	1991	
		Value of sales		
First		8	20	40
Second		30	50	62
Third		60	80	92
Fourth	24	20	40	

a. Using an additive model, find the centred moving average trend.

b. Find the average seasonal variation for each quarter.

c. Predict sales for the last quarter of 1992 and the first quarter of 1993

Solution
1.

Year	Quarter	(i) Sales	(ii) 4-point Moving Average	(iii) Centred Moving Average	(iv) (i) - (iii)
1989	4	24			
1990	1	8			
			30.5		
	2	30		30	0
			29.5		
	3	60		31	29
			32.5		
	4	20		35	-15

			37.5		
1991	1	20		40	-20
			42.5		
	2	50		45	5
			47.5		
	3	80		50	30
			52.5		
	4	40		54	-14
			55.5		
1992	1	40		57	-17
			58.5		
	2	62			
	3	92			

2. The difference between the actual sales and the trend is in column (iv) of the table. By averaging the appropriate quarterly values, an estimate of the 'seasonal' factors may be obtained.

Quarter	1	½(-20+(-17))	=	-18.5
Quarter	2	½(0+5)	=	2.5
Quarter	3	½(29+30)	=	29.5
Quarter	4	½(-15+(-14))	=	-14.5

As these four values do not sum to zero (as they should using an additive model), they may be corrected by adding 0.25 to each value to give:

-18.25; 2.75; 29.75; -14.25

3. An upward trend in the data is obvious and to predict future sales, this trend must be extrapolated. The trend does not seem to be completely regular and without more complicated analysis (e.g. fitting a linear trend to the data by a process of least

squares), the average quarterly increase in the trend seems to be approximately

4. On the basis of the last trend value (57) the predicted trend should be:

1992	-	Quarter 4:	57 + 3 x 4 = 69
1993	-	Quarter 1:	57 + 4 x 4 = 73

This gives actual sales estimates of:

69 - 14.25 = 54.75 i.e 55

and 73 - 18.25 = 54.75 i.e. 55

Example 2

The following data are the production of coca ('000 tons) in Nigeria for the year 1988 to 2002:

YEAR	1988	1989	1990	1991	1992	1993	1994
Production ('000)	4	3	4	5	9	9	15
YEAR	1996	1997	1998	1999	2000	2001	2002
Production ('000)	10	26	17	18	35	24	40

a. Using the least square method, fit the linear trend
   $Y = a + bT$, where $T = t - 8$
b. Calculate the trend value for each year.
c. Predict the production for 2003, using the fitted trend.

Solution

a.

T	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
Production(Y) ('000)	4	3	4	5	9	9	15	10	26	17	18	31	35	24	40
T^2	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49
TY	-28	-18	-20	-20	-27	-18	15	0	26	34	54	124	175	144	280

$\sum T = 0$, $\sum Y=260$, $\sum T^2=280$, $\sum TY=751$

$\hat{a} = 260/15 = 17.30$, $b = 751/280 = 2.68$

The required equation is

$$Y = 17.3 + 2.7T$$
$$= -4.3 + 2.7t$$

b. The trend values are

Year Trend Values

1988	-1.6
1989	1.1
1990	3.8
1991	6.5
1992	9.2
1993	11.9
1994	14.6
1995	17.3
1996	20.0
1997	22.7
1998	25.4
1999	28.1
2000	30.8
2001	33.5
2002	36.2

c. Production at $t = 16$, 2003 is $Y = 49.7$ tons.

Summary

In this study session you have learnt about:

1. **Define Forecasting**

   A planning tool that helps management in its attempts to cope with the uncertainty of the future, relying mainlly on data from the past and present and analysis of trends. Forecasting starts with certain assumptions based on the management's experience, knowledge, and judgment.

2. **Elementary Forecasting**

   Very often, we have a series of measurements, which are affected by the same time series components, such as trend and seasonal variation; these are accounted for in forecasting. Any change in the more sensitive series will anticipate the corresponding change in the components therein, and can be used as a forecasting indicator.

   Generally, the forecasting for period p is

   a. $X_p = T_p \text{ x } SI_p$ and

   b. $X_p = T_p + SI_p$ for both the multiplication and additive models respectively.

Self-Assessment Questions (SAQs) for study session 15

Now that you have completed this study session, you can assess how well you have achieved its Learning outcomes by answering the following questions. Write your answers in your study Diary and discuss them with your Tutor at the next study Support Meeting. You can check your Define School answers with the Notes on the Self-Assessment questions at the end of this Module.

SAQ 15.1 (Testing Learning Outcomes 15.1)

What is forecasting

SAQ 15.2 (Testing Learning Outcomes 15.2)

Explain Elementary Forecasting

References

John, E. F.(1974). Modern Elementary Statistics, International Edition. London: Prentice Hall

Murray, R. S. (1972) Schaum's Outline Series. Theory and Problems of Statistics. New York: McGraw-Hill Book Company

Shangodoyin, D. K. and Agunbiade D. A. (1999). Fundamentals of Statistics Ibadan: Rasmed Publications.

Shangodoyin, D. K. and Ojo, J. F. (200). Elements of Time Series Analysis Ibadan: Rasmed Publications.

http://www.businessdictionary.com/definition/forecasting.html

Notes on SAQ

Study Session 1

1.1 Branches of mathematics concerned with collection, classification, analysis, and interpretation of numerical facts, for drawing inferences on the basis of their quantifiable likelihood

1.2 Types of Statistics

- Descriptive Statistics
- Statistics method
- Statistical inference

1.3 Population: Population is a collection of the individual items, whether of people or thing, that are to be observed in a given problem situation.

Study Session 2

2.1 Sampling is a method of selecting a subset or part of a population that is representative of the entire population.

2.2 Some of the Non-probability sampling techniques are:

1. **Quota Sampling:** This is one where, although the population is divided into identified groups, elements are selected from each group without recourse to randomness. Here the interviewer is free to use his discretion to select the units to be included in the sample. This method is commonly used in opinion poll, by the journalist and in market research.
2. **Judgmental or Purposive Sampling:** This is a sample whose elementary units are chosen according to the discretion of expert who is familiar with the relevant characteristics of the population. These sampling units are selected judgmentally, and there is a heavy possibility of biasness.

2.3 (a) Population

The collection of all units of a specified type in a given region at a particular point or period of time is termed as a population or universe. Thus, we may consider a population of persons, families, farms, cattle in a region or a population of trees or birds in a forest or a population of fish in a tank etc. depending on the nature of data required.

(b) Sampling Unit

Elementary units or group of such units which besides being clearly defined, identifiable and observable, are convenient for purpose of sampling are called sampling units. For instance, in a family budget enquiry, usually a family is considered as the sampling unit since it is found to be convenient for sampling and for ascertaining the required information. In a crop survey, a farm or a group of farms owned or operated by a household may be considered as the sampling unit.

Study Session 3

3.1 The Central Limit Theorem provides us with a shortcut to the information required for constructing a sampling distribution.

By applying the Theorem we can obtain the descriptive values for a sampling distribution (usually, the mean and the standard error, which is computed from the sampling variance) and we can also obtain probabilities associated with any of the sample means in the sampling distribution.

3.2 The Variance of the Sampling Distribution of Means: Parameter Known

According to the Theorem, the variance of the sampling distribution of means equals the population variance divided by N, the sample size. The population variance (ó) and the size of the samples (N) drawn from that population have been identified in the preceding chapter as the two key factors which influence the variability of the sample means.

3.3 Large Sample Distribution of Means

Suppose a random sample of size n_A from population A yield a mean $\overline{X}_A$; we know that provided n_A is large and sampling is done from a finite population and it is done with replacement then $\overline{X}_A \sim N(\mu_A, \sigma_A^2 / n_A)$ and

$$Z = \frac{\overline{X}_A - \mu_A}{\sqrt{\frac{\sigma_A^2}{n_A}}}$$

which has the standard normal distribution; μ and σ being population mean and standard deviation respectively. When the population variance σ^2 is not known but n is sufficiently large, we may use $\hat{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ in its place.

Study Session 4

4.1 Large Sample Distribution of Proportion

The mean of the distribution of sample proportions is equal to the population proportion, p .

If p is unknown, we estimate it using p ^ .

4.2 Large Sample Distribution of Difference of Proportions

Statistics problems often involve comparisons between two independent sample proportions. This lesson explains how to compute probabilities associated with differences between proportions.

Study Session 5

5.1 Definitions of Estimation

Estimation is a process by which a statistic (summary of a collection of numerical data, e.g. total, range, average, etc.) obtained from a sample is used to estimate the parameters of the population from which the sample has been drawn. Its need arises in practically every statistical decision-making in all spheres of life. The following are the nature of estimates, we are bound to encounter in everyday usage.

5.2 Confidence Interval

The only difference between calculating the interval for percentages or for proportions is that the former total 100 and the latter total 1. This difference is reflected in the formulae used, otherwise the methods are identical.

Study Session 6

6.1 Large Sample Interval Estimation for Mean

If the statistic S is the sample mean $\overline{x}$, then 95% and 99% confidence level for estimation of the population mean μ, are given by $\overline{x} \pm 1.96\sigma_{\overline{x}}$ and $\overline{x} \pm 2.58\sigma_{\overline{x}}$ respectively. Generally, the confidence limits are given as $\overline{x} \pm Z_i \sigma_x$, where Zi is the level of confidence desired and can be got from the table. The sample variance is then given as $\frac{\sigma^2}{n}$. Thus, the confidence interval for the population mean is then given as $\overline{x} \pm Z_c \frac{\sigma}{\sqrt{n}}$.

6.2 Large Sample Estimation of a Population Mean

The Central Limit Theorem says that, for large samples (samples of size $n \geq 30$), when viewed as a random variable the sample mean *X*−− is normally distributed with mean *μX*−−=*μ* and standard deviation *σX*−−=*σ*/*n*−−√. The Empirical Rule says that we must go about two standard deviations from the mean to capture 95% of the values of *X*−− generated by sample after sample.