MAT 241
Ordinary Differential Equation

Ibadan Distance Learning Centre Series

MAT 241
Ordinary Differential Equation

By
H.O. Tejumola
Department of Mathematics
University of Ibadan

Published by
Distance Learning Centre
University of Ibadan

First published 1993

Reprint 2007

ISBN 978-021-041-5

General Editor: Prof. Francis O. Egbokhare
Series Editor: Mrs. O. I. Adeyemo and Mr. O.O. Akisanya

Typeset @ Distance Learning Centre U.I. Ibadan

Printed by: University of Ibadan Printery, Atiba Road Second Gate, U.I., Ibadan.

Table of Content

			Page
Vice-Chancellor's Message		… … … … …	vi
Foreword		… … … … … … …	vii
Lecture One	:	How Ordinary Differential Equations Arise	1
Leture Two	:	Definition and Classification	5
Lecture Three	:	Variable Separables Equation	10
Lecture Four	:	Equations Reducible to Variable Separable	14
Lecture Five	:	Linear Equations	21
Lecture Six	:	Exact Equations	27
Lecture Seven	:	Some Special Equations	34
Lecture Eight	:	Second and Higher Order Equations with Constant Co- efficients	37
Lecture Nine	:	Particular Integrals	43
Lecture Ten	:	Initial-Value Problem	49
Lecture Eleven	:	Equations with Variable Co-efficients	52
Lecture Twelve	:	Higher Order Equations	57
Lecture Thirteen	:	Difference Equations	63
Lecture Fourteen	:	Properties of Solutions	68
Lecture Fifiteen	:	Non-honogenous Equation with Constant Co- efficient	73

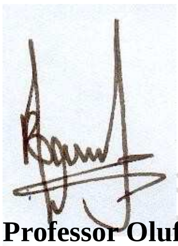
Vice-Chancellor's Message

I congratulate you on being part of the historic evolution of our Centre for External Studies into a Distance Learning Centre. The reinvigorated Centre, is building on a solid tradition of nearly twenty years of service to the Nigerian community in providing higher education to those who had hitherto been unable to benefit from it.

Distance Learning requires an environment in which learners themselves actively participate in constructing their own knowledge. They need to be able to access and interpret existing knowledge and in the process, become autonomous learners.

Consequently, our major goal is to provide full multi media mode of teaching/learning in which you will use not only print but also video, audio and electronic learning materials.

To this end, we have run two intensive workshops to produce a fresh batch of course materials in order to increase substantially the number of texts available to you. The authors made great efforts to include the latest information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly. It is our hope that you will put them to the best use.



Professor Olufemi A. Bamiro, FNSE
Vice-Chancellor

Foreword

The University of Ibadan Distance Learning Programme has a vision of providing lifelong education for Nigerian citizens who for a variety of reasons have opted for the Distance Learning mode. In this way, it aims at democratizing education by ensuring access and equity.

The U.I. experience in Distance Learning dates back to 1988 when the Centre for External Studies was established to cater mainly for upgrading the knowledge and skills of NCE teachers to a Bachelors degree in Education. Since then, it has gathered considerable experience in preparing and producing course materials for its programmes. The recent expansion of the programme to cover Agriculture and the need to review the existing materials have necessitated an accelerated process of course materials production. To this end, one major workshop was held in December 2006 which have resulted in a substantial increase in the number of course materials. The writing of the courses by a team of experts and rigorous peer review have ensured the maintenance of the University's high standards. The approach is not only to emphasize cognitive knowledge but also skills and humane values which are at the core of education, even in an ICT age.

The materials have had the input of experienced editors and illustrators who have ensured that they are accurate, current and learner friendly. They are specially written with distance learners in mind, since such people can often feel isolated from the community of learners. Adequate supplementary reading materials as well as other information sources are suggested in the course materials.

The Distance Learning Centre also envisages that regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks will find these books very useful. We are therefore delighted to present these new titles to both our Distance Learning students and the University's regular students. We are confident that the books will be an invaluable resource to them.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.

Professor Francis O. Egbokhare
Director

LECTURE ONE

How Ordinary Differential Equations Arise

Introduction

Many of the basic laws of the physical sciences and, more recently, of the biological and social sciences, are formulated in terms of mathematical relations involving certain known and unknown quantities and their rate of change (**derivatives**). Such relations are called **differential equations.** To further illustrate these, let us consider a few examples.

Example 1.1

A boy drops a ball from the top of a building. 22.05 metres high. When will the ball hit the ground?

Solution

Assume for simplicity that, the ball is a point mass, and that there is no air resistance. Then the movement of the ball is duly only to the force of gravity. If h(t) denotes the height of the ball at any time (t), then its velocitv. which the instantaneous rate of change of height(h) with respect to time, is $h'(t)$. Similarly, its acceleration, which is the rate of change of velocity $h'(t)$ with respect to time, is $h''(t)$. Since the gravitational acceleration at the surface of the earth is 930cm/sec^2 (980 centimetre per second per second). = 32.2ft/sec^2, it follows that $h''(t) = -980$ ——1.1

That is, the acceleration of the ball is constant and is in a direction opposite to gravitational acceleration, which is measured downwards. (This is why there is a negative sign in 1.1.

If you now integrate both sides of 1.1 with respect to t, you will obtain

$$h(t) = -980t + C_1 \qquad \text{——1.2}$$

where C_1 is a constant of integration. To determine C_1, set $t = 0$ in 1.2 and observe that the initial velocity of the ball is $\dot{h}(o)$ and is zero. Therefore $C_1 = h(o) = 0$. To obtain an expression for h(t), integrate 1.2 with respect to t, to obtain

$$h(t) = -490t^2 + C_2 \qquad \text{——1.3}$$

with C_2 as a constant of integration. Setting $t = 0$ in 1.3 gives

$$C_2 = h(0) = 2205.$$

Since the initial height of the ball above the ground is the height of the building itself. With this value of C_2, 1.3 becomes

$$h(t) = -490t^2 + 2205, \qquad \text{——1.4}$$

and this relation gives the height of the ball above the ground at any time t. When the ball hits the ground $h(t) = 0$; that is,

$$490t^2 = 2205$$

Therefore $t = \pm 3/\sqrt{2}$. Since $t = -3/\sqrt{2}$ has no physical meaning, the desired answer is $t = 3/\sqrt{2}$ seconds.

Example 1.2

Consider a bacteria population that is changing at a rate proportional to its size. If $N = N(t)$ represents the population at time t, then dN/dt is rate of change of population, and $dN/dt \ \alpha \ N$; that is

$$dN/dt = kN \qquad \text{——1.5}$$

is the equation of population growth, where k is the constant of proportionality.

Solution

A solution of this equation is $N(t) = N(0)e^{kt}$.

You will see that if $K>0$, then the population is growing exponentially, while if $K<0$, the population will decay exponentially. If $K = 0$ the population will remain constant at the initial value P(0).

Example 1.3

Newton's law of cooling states that the rate of change of the temperature, difference between an object and its surrounding medium is proportional to the temperature difference. You will see that if you denote the temperature difference at time t by $\Delta(t)$, then the equation for $\Delta(t)$ is the same as 1.5; that is,

$$\frac{d\Delta}{dt} = k\Delta$$

with solution

$$\Delta(t) = \Delta(0)\, e^{kt}$$

since the temperature difference approaches zero as time increases without bound, the constant k in this case must be negative.

Example 1.4

The growth rate per individual in a population is the difference between the average birth rate and the average death rate. Suppose that in a given population the average birth rate is a constant $\beta > 0$, but the average death rate, due to effects of over-crowding and increased competition for limited food resources, is proportional to the size of population, with constant of proportionality $\delta > 0$.

Solution

Let P = P(t) be the population size at time t. Then dP/dt is the growth of the population and the growth of the population and the growth rate per individual of the population is

$$1/p\ dP/\ dt.$$

The birth rate per individual is β and the death rate is $-\delta P$. Therefore

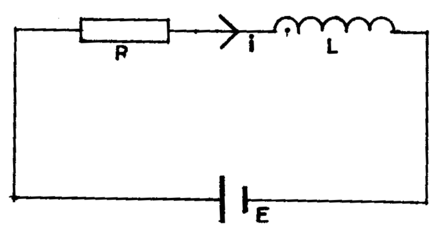
$$1/p\ dP/dt = \beta - \delta P \qquad \text{——1.6}$$

and this is called the **logistic equation**. The growth exhibited by this equation is called the **logistic growth**. You will see in the next section (Example) that

$$P(t) = \beta/\delta + [\beta p^{-1}(0) - \delta]e^{-\beta t} \text{ is a solution of 1.6.}$$

Observe now that as t gets larger, the term $e^{-\beta t}$ approaches zero (since $\beta > 0$) and the population size approaches a limiting value of β/δ beyond which it cannot increase, since setting $P = \beta/\delta$ in 1.6 yields dP/dt = 0.

Example 1.5



The circuit shown above gives rise to

$$L\frac{di}{dt} + Ri = E \qquad \text{——1.7}$$

where i is the current at time, t.

Example 1.6

A family, S of curves is such that the gradient at every point of every member of the family is twice the abscissa. Determine the family of curves.

Solution

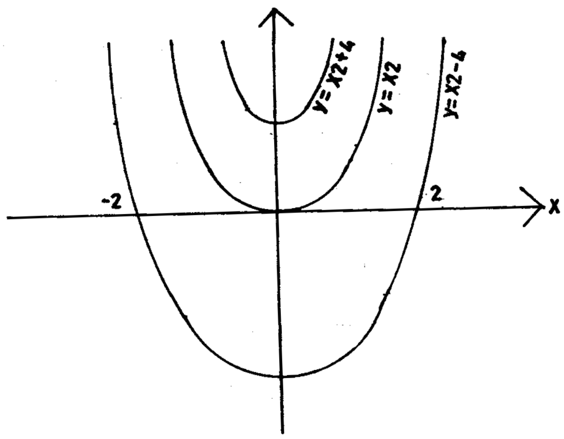
Let (x,y) be an arbitrary point of one member of S. Then dy/dx is the gradient and

$$\frac{dy}{dx} = 2x \qquad \text{——1.8}$$

Integrating both sides with respect to x gives

$$y = x^2 + C \qquad \text{——1.9}$$

where C is a constant. This is the equation of a family of parabolas with common axis y.



Note

In any of the above, the problem is to find the dependent variable in terms of the independent one, e.g. in 1.5 N in terms of t, while 1.8 y in terms of x.

LECTURE TWO

Definition and Classification

Introduction

A relationship between a variable quantity x and a dependent variable y and its derivatives

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \text{.........}$$ is called an **Ordinary Differential Equation.**

Some examples are:

$$\frac{dy}{dx} = bx \qquad \text{—— 2.1}$$

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = 0 \qquad \text{—— 2.2}$$

$$\frac{dy}{dx^2} + 6y = 0 \qquad \text{—— 2.3}$$

$$\frac{x^3d^3y}{dx^3} + \frac{x^2d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 \qquad \text{—— 2.4}$$

The order of a differential equation is the order of the highest derivative that appears in the equation. Thus, you will see that equations 2.1, 2.2 and 2.4 are of orders 1, 2 and 3 respectively.

The degree of a differential equation is the power to which the highest derivative is raised. Clearly equations 2.1 and 2.2 are of degree 1 while 2.3 and 2.4 are of degree 2.

As you have seen in Lecture one differential equations normally arise from physical situations and it is often required to obtain a functional relationship between the independent variable and the dependent variable, having eliminated the derivatives. For example, equation 1.9 gives such a functional relationship for the equation 1.8. This relation is referred as the SOLUTION of differential equation.

Example 2.1

Consider the equation $\frac{d^2y}{dx^2} = x^2$ —2.5

which is of first of second order and first degree. You can solve this equation by two successive integrations:

$$\frac{dy}{dx} = \frac{1}{3}x^3 + A$$

$$\text{and } y = \frac{1}{12}x^4 + Ax + B, \qquad \text{—2.6}$$

where A, B are arbitrary constants.

You will note from this example and 1.8 that the solutions of the first and second order equations involve one and two arbitrary constants respectively. In general it can be shown that the solution of an ordinary differential equation referred to henceforth as o.d.e., of order n involves n arbitrary constants.

Conversely, it is known that if a functional relationship between y and x contains n arbitrary constants, then the elimination of these constants yields an o.d.e. of order n.

Example 2.2

Consider the relationship

Solution

$$y = Ae^{2x} + Be^{x}.$$

Differentiating with respect to x two times gives

$$y' = 2Ae^{2x} + Be^{x}$$

and

$$y'' = 4Ae^{2x} + Be^{x}$$

The first two equations give

$y'' - 3y' = -2Ae^{2x} - 2Be^{x}$.

But, $y = Ae^{2x} + Be^{x}$. Therefore

$y'' - 3y' = -2Ae^{2x} - 2Be^{x} = -2y$;

that is,

$$y'' - 3y' + 2y = 0,$$

which is an o.d.e. of order 2.

Solutions, such as 1.9 and 2.6, with the appropriate number of arbitrary constants are called **General Solutions.**

In physical problems, solutions are usually required which satisfy certain specified conditions, and these conditions are used to assign values to the arbitrary constants. This type of solution, that satisfies certain given conditions, is called a **Particular Solution**, and the conditions satisfied are of two types: **Boundary Conditions** or **Initial Conditions.**

Example 2.3

Now return to equation 1.8 and its general solution 1.9. Let the initial condition be $y = 1$ when $x = 0$. Then you will see readily that the value assigned to the arbitrary constant C in this case is obviously $C = 1$, and the particular solution is $y = x^2 + 1$

Example 2.4

You have seen from example 2.2 that $y = Ae^{2x} + Be^{x}$ is a solution of $y'' - 3y' + 2y = 0$.

Furthermore it is the general solution since it contains two arbitrary constants. Now let the value of y for a given value of x thus be $y = 0$ where $x = 0$. This gives $A + B = 0$, which clearly does not determine A, B independently. We need a second condition in order to do this. Let the conditions now be

$$y(0) = 0, \quad y'(0) = 1. \qquad \text{—— 2.7}$$

You can readily show that these lead to two equations in A and B,

$$A + B = 0, 2A + B = 1$$

and hence that $A = 1$, $B = -1$. The particular solution is $y = e^{2x} - e^{x}$ and 2.6 are the initial conditions.

In general if the o.d.e. concerned is of order n, then the initial conditions consist of specifying the values of the dependent variable together with those of first (n-1) derivatives at a given value of the independent variable; for example

$$y(0) = a_0, y^{i}(0) = a_i, i = 1, 2, \text{..........} n - 1$$

If instead of 2.7, the values

$$y'' - 3y' = -2Ae^{2x} - 2Be^{x}.$$

But, $y = Ae^{2x} + Be^{x}$. Therefore

$$y'' - 3y' = -2Ae^{2x} - 2Be^{x} = -2y;$$

that is,

$$y'' - 3y' + 2y = 0,$$

which is an o.d.e. of order 2.

Solutions, such as 1.9 and 2.6, with the appropriate number of arbitrary constants are called **General Solutions.**

In physical problems, solutions are usually required which satisfy certain specified conditions, and these conditions are used to assign values to the arbitrary constants. This type of solution, that satisfies certain given conditions, is called a **Particular Solution**, and the conditions satisfied are of two types: **Boundary Conditions** or **Initial Conditions.**

Example 2.3

Now return to equation 1.8 and its general solution 1.9. Let the initial condition be $y = 1$ when $x = 0$. Then you will see readily that the value assigned to the arbitrary constant C in this case is obviously $C = 1$, and the particular solution is $y = x^2 + 1$

Example 2.4

You have seen from example 2.2 that $y = Ae^{2x} + Be^{x}$ is a solution of $y'' - 3y' + 2y = 0$.

Furthermore it is the general solution since it contains two arbitrary constants. Now let the value of y for a given value of x thus be $y = 0$ where $x = 0$. This gives $A + B = 0$, which clearly does not determine A, B independently. We need a second condition in order to do this. Let the conditions now be

$$y(0) = 0, \quad y'(0) = 1. \qquad \text{—— 2.7}$$

You can readily show that these lead to two equations in A and B,

$$A + B = 0, \; 2A + B = 1$$

and hence that $A = 1$, $B = -1$. The particular solution is $y = e^{2x} - e^{x}$ and 2.6 are the initial conditions.

In general if the o.d.e. concerned is of order n, then the initial conditions consist of specifying the values of the dependent variable together with those of first (n-1) derivatives at a given value of the independent variable; for example

$$y(0) = a_0, \; y^{i}(0) = a_i, \; i = 1, 2, \; \ldots\ldots\ldots n - 1$$

If instead of 2.7, the values

$$y(0) = 0, \; y(1) = 1 \qquad \text{—— 2.8}$$

are specified, then 2.8 are called boundary values and, in this case the arbitrary constants must be determined from

$$A + B = 0, \; Ae^2 + Be = 1.$$

You will see from these equations that $B = e^{-2}/(e^{-1} - 1)$ and $A = -e^{-2}/(e^{-1} - 1)$; and hence that $Y = e^{-2}/(e^{-1} - 1)\,[e^x - e^{2x}]$ is the desired particular solution.

Post-Test

1. Solve the following differential equations by integrating both sides. Obtain the particular solution satisfying the given initial conditions

(a) $\frac{dy}{dx} = 2x^3 + 3, \; y(0) = 2$

(b) $\frac{dy}{dx} = \sin 2x$

(c) $\frac{dy}{dx} = \frac{2X}{x^2 + 5}, \; y(1) = 4$

(d) $(1 - x^2)\frac{dy}{dx} = 1$

(e) $\frac{d^2y}{dx^2} = \sin x - \cos x \; y(\pi/2) = 0, \; y'(\pi/2) = 1$

(f) $\frac{d^2y}{dx^2} = x^2 + a, \qquad y(0) = 1, \; y'(0) = 2$

(g) $e^{2x}\frac{dy}{dx} + 1 = 0 \qquad y \to 1/2 \text{ as } x \to +\infty$

2. Solve the following equations and the systems of curves represented by the general solutions:

(a) $\frac{dy}{dx} = \frac{1}{3}X$

(b) $\frac{dy}{dx} = e^x$

(c) $\frac{dy}{dx} = \text{Sin } x$

(d) $x^2\frac{dy}{dx} + 2 = 0$

LECTURE THREE

Variable Separables Equation

Introduction

You now have the necessary background for the study of solutions of ordinary differential equations. You have, indeed, solved some o.d.e's by direct integration. However, there are many other equations which cannot be solved that way. In order to solve such equations, you will need some additional techniques. These new techniques will be introduced in what follows for first order equations.

First order o.d.e's can be classified into four broad headings:

(i) Variable Separable Equation.

(ii) Equation Reducible to Variable Separable form

(iii) Linear Equations

(iv) Exact Equations

Consider the equation

$$\frac{dy}{dx} = F(x,y). \qquad \text{—— 3.1}$$

The simplest situation when this equation can be solved is when F(x,y) is a product of the form

$$F(x,y) = f(x)\, g(y)$$

In such a case, equation 3.1 becomes

$$\frac{dy}{dx} = f(x)\,g(y) \qquad \text{—— 3.2}$$

We may then "separate the variables" and write

$$\frac{dy}{g(y)} = f(x)dx.$$

The general solution is obtained by direct integration thus:

$$\int \frac{dy}{g(y)} = \int f(x)dx + A,$$

Where A is an arbitrary constant of integration. Equations of the form

$\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = g(y)$ are specially simple examples of 3.2, and problems 1 and 2 at the end of Lecture 2 concern such equations.

Example 3.1

Solve the equation_

$$\frac{dy}{dx} = (x^3 + 1)\,y^2$$

Solution

To solve this problem, first express the equation in the form 3.3 and then integrate to give:____

$$\frac{dy}{y^2} = (x^3 + 1)\,dx;$$

that is,

$$-y^{-1} = 1/4x^4 + x + A,$$

where A is an arbitrary constant.

Example 3.2

Solve the equation

$$x(y^2 - 1)dx - y(x^2 - 1)dy = 0$$

Solution

Separating the variables gives:

$$\frac{xdx}{x^2 - 1} - \frac{ydy}{y^2 - 1} = 0$$

To integrate, you must recall that $d/dx(x^2 - 1) = 2x$. Therefore, on integrating, you will obtain:

$$1/2 \log |x^2 - 1| - 1/2 \log |y^2 - 1| = A$$

or

$$\log | x^2 - 1/y^2 - 1| = -\log B, \qquad A = -\log B,$$

which may be rewritten thus:

$$(y^2 - 1) = B(x^2 - 1).$$

where B is an arbitrary constant.

Observe that replacing one arbitrary constant by another, as in this case $A = -\log B$; may help simplify the expression for the general solution.

Example 3.3

Solve the logistic equation

$$\frac{dp}{dt} = p(\beta - \delta P)$$

Solution

Separating variables, you have that

$$\frac{dp}{p(\beta - \delta p)} = dt.$$

By partial fraction: you also have that

$$\frac{1}{P(\beta - \delta p)} = \frac{1}{\beta P} + \frac{\delta}{\beta(\beta - \delta P)},$$

and substitution into the separated equation and integration now give

$$\frac{1}{\beta} \ln P - \frac{1}{\beta} \ln (\beta - \delta P) = t + A$$

$$\text{or } \ln \left(\frac{P}{(\beta - \delta P)}\right)^{1/\beta} = t + A.$$

You may take the exponential of both sides and replace e^A by A to obtain

$$\frac{P}{\beta - \delta P} = Ae^{\beta t}$$

as the general solution.

If the initial condition

$$P(0) = P_0$$

is prescribed, the arbitrary constant A can be eliminated. Indeed

$$A = P_0/(\beta - \delta P_0)$$

and so

$$\frac{P(t)}{\beta - \delta P(t)} = \frac{P_0}{\beta - \delta P_0} e^{\beta t}$$

If you now cross multiply and solving for P(t), you will obtain (after some algebra)

$$P(t) = \frac{\beta}{\delta + [\beta P_{0}^{-1} - \delta]\, e^{-\beta t}},$$

which is the solution of the logistic equation.

Post-Test

1. Decide which of the following equations is of variable separable type, and separate the variables where possible:

(a) $\frac{dy}{dx} = \cot x \cot y$

(b) $(x^2 + 1)\, dy + (xy - xy^2)\, dx = 0$

(c) $xydy + \sqrt{(1 - y^2)}\, dx = 0$

(d) $(1 - x) \frac{dy}{dx} + (xy + \sin x) = 0$

(e) $\sin y \frac{dy}{dx} + \sin x + y^2 = 0$

(f) $\frac{dy}{dx} + (1 - y^2) + \tan x = 0$

2. If an equation in problem 1 is separable, solve it.

3. Solve the following initial value problems:

(a) $\frac{dy}{dt} = 3y, \quad y = 1$ when $t = \frac{1}{3}$

(b) $\frac{dy}{dx} = \sec 3y, \quad y = \frac{\pi}{6}$ when $x = 0$

(c) $x \frac{dy}{dx} = y + xy, \; y = 1$ when $x = 1$

(d) $2z(x + 1) \frac{dz}{dx} = 4 + z^2, \; z = 2$ when $x = 3$

4. Solve the following

(a) $(1 + x^2)^2 \frac{dy}{dx} + xy^2 = 0, \quad y \to + \infty$ as $\infty \to 0$

(b) If $x(x=1) \frac{dy}{dx} = y(y+1)$ and $y = 2$ when $x = 1$, find y when $x = 2$

Equations Reducible to Variable Separable

Introduction

It is sometimes possible to reduce an equation to the variable separable type by means of a simple change in variable. Consider

$$\frac{dy}{dx} = \cos(x + y), \qquad \text{—— 4.1}$$

in which the variables are not separable. Make the substitution $x + y = v$. Then, you will have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}.$$

If you substitute for dy/dx in the given equation 4.1, you will obtain

$$\frac{dv}{dx} - 1 = \cos v$$

and this separates to give

$$\frac{dv}{1 + \cos v} = dx$$

Note now that $1 + \cos v = 2\cos^2(v/2)$. Therefore on integrating you will obtain

$$\tan\frac{v}{2} = x + A$$

or

$$y = 2\tan^{-1}(x + A) - x$$

Pre-Test

Solve the following equations:

(a) $(x + y)\,dx + dy = 0$

(b) $\frac{dy}{dx} = \tan^2(x + y)$

Another instance of reduction by change of variables occurs in the case of homogeneous equations, which now follows.

Homogeneous Equations

A function $P(x,y)$ is said to be homogeneous of degree n if the sum of the powers of x and y in each term of P is n. For example

(i) $xy^2 - 3x^2y = 2x^3$ is homogeneous of degree 3 and

(ii) $y^4 - 3x^2y^2 + 2xy^3$ is homogeneous of degree 4

Now if in (i) you set

$$P(x,y) = xy^2 - 3x^2y + 2x^3$$

and let $y = vx$. Then you will see readily that

$$P(x,vx) = x(vx)^2 - 3x^2(vx) + 2x^3$$
$$= x^3[v^2 - 3v + 2]$$

Similarly (ii) will give

$$P(x,vx) = x^4[v^4 - 3v^2 + 2v^3]$$

More generally if $P(x,y)$ is a homogeneous function of degree n, the substitution $y = vx$ will reduce P to the form

$$P(x, xv) = x^nR(v)$$

where R is independent of x.

Consider now the equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \qquad \text{—— 4.2}$$

This equation is said to be of homogeneous functions of the same degree. If both P, Q are homogeneous of degree n, the substitution $y = vx$ will reduce them to the forms

$P(x, xv) = x^nR(v)$, $Q(x, xv) = x^n S(v)$ where R and S are functions of v alone. Substituting $y = vx$ in 4.2, noting that $dy = xdv + vdx$ and cancelling out x^n, you will obtain

$$R(v)dx + S(v)\{vdx + xdv\} = 0$$
$$\text{or } \{R(v) + vS(v)\}\,dx + xS(v)\,dv = 0$$

Separating variables and integrating now gives

$$\int \frac{s(v)dv}{R(v) + vS(v)} + \log x = A$$

When you evaluate this integral and substitute (back) $v = y/x$, you will obtain the general solution.

Example 4.1

Which of the following are homogeneous:

(a) $(x^2 + y^2)\frac{dy}{dx} = xy$

(b) $(x^2 + y)\frac{dy}{dx} = xy$

(c) $(x^2 - y^2)\,dx + 2xy\,dy = 0$

(d) $\frac{dy}{dx} = y(3x^2 + y^2)/x(x + 3y)$

(e) $c(2x + y)\,dy - y\,(x+2y)\,dx = 0$

Solution

(a), (c) and (e) are homogeneous. (b) and (d) are not homogeneous. In (b) $x^2 + y$ is not homogeneous, while in (d) both numerator and denominator are homogeneous, but of different degrees 3 and 2 respectively.

Example 4.2

Solve:

$$(x^2 + y^2)\frac{dy}{dx} = xy$$

Solution

The given equation is the same as

$$xydx - (x^2 + y^2)\,dy = 0$$

Both xy and $x^2 + y^2$ are homogeneous of degree 2. Substituting $y = vx$, you will obtain

$$x^2vdx - (1 + v^2)x^2(vdx + xdv) = 0$$

and, if x^2, is cancelled, then

$$v^3dx + x(1+v^2)dv = 0.$$

Now separate variables and integrate to obtain

$$\int dx/x + \int \frac{1 + v^2}{v^3dv} = A$$

$$\text{or} - \frac{1}{2v^2} + \log(vx) = A$$

Therefore the solution is

$$\log \frac{y}{A} = \frac{x^2}{2y^2}$$

Example 4.3

Solve: $$\frac{dy}{dx} = \frac{y(3x^2 + y^2)}{x(x^2 + 3y^2)}$$

Solution

Here both numerator and denominator are homogeneous of degree 3. Divide both numerator and denominator by x^3 to obtain

$$\frac{dy}{dx} = \frac{3x^2y + y^3}{x^3 + 3xy^2} = \frac{3y/x + (y/x)^3}{1 + 3\ (y/x)^2}$$

Setting $y = vx$ or $y/x = v$, you will obtain

$$v + x\frac{dv}{dx} = \frac{3v + v^3}{1 + 3v^2}$$

$$\text{or } \frac{dx}{x} = \frac{1 + 3v^2}{2v1\text{-}v^2)}\ dv.$$

Lastly if you split this into partial fractions you will obtain

$$\frac{dx}{x} = \frac{1}{2}\ \frac{1}{[v} + \frac{2}{1 - v} - \frac{2}{1 + v]}\ dv$$

and integration now gives

$$\log x^2 = \log Av/(1 - v^2)^2;$$

that is

$$(x^2 - y^2)^2 = Axy$$

Equations with linear Co-efficient

Although the equation

$$(ax + b_1y + c_1)\ dx + (a_2x + b_2y + c_1)\ dy = 0 \qquad \text{—— 4.3}$$

is not of the homogeneous type, it may be reduced to that type by a change o variable. This can be shown thus.

The equations

$$a_1x + by_1 + C_1 = 0,\ a_2x + b_2y + c_2 = 0 \qquad \text{—— 4.4}$$

represent straight lines, which will intersect unless the condition

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \qquad \text{—— 4.5}$$

for them to be parallel hold. Now let $a_2/a_1 \neq b_2/b_1$ and let the lines intersect at (h, k). Transfer the origin to that point by the substitution

$$x = h + x, y = k + y,$$

and the equation 4.3 becomes

$$(a_1x + b_1y)\, dx + (a_2x + b_2y)dy = 0.$$

This is of the homogeneous type. You may now substitute y = vx, separate variables and obtain the general solution:

$$\log Ax + \int \frac{(a_2 + b_2v)dv}{a_1 + (a_2 + b_1)\, v + b_2v^2}$$

The final form of the solution will depend on whether the roots of the denominator in the integrand are real, coincident or complex; that is according as $(a_2 + b_1)^2$ is greater than, equal to, or less than $4a\ b_2$.

In the exceptional case when the lines 4.4 are parallel, so that by 4.5,

$\frac{a_1}{a2} = \frac{b_1}{b2} = \frac{1}{k}$, you may rewrite 4.3 in the form

$$(a'x + b_1y + c_1)\, dx + \{k(a'x + b'y) + c_2\}dy = 0$$

Now choose $z = a'x + b_1y$ as the new variable, then

$$b_1(z + c_1)\, dx + (kz + c_2)\,(d_2 - a_1\, dx) = 0$$

Separating variables and integrating you will obtain

$$x + \int \frac{(kz + c2)dz}{(b_1 - a_1k)\, t + b_1c_1 - ac_2} = \text{constant.}$$

as the general solution, with $z = a'x + b'y$.

Example 4.4

Solve the equation

$$(x - y + 4)dx - (x + y + 2)\, dy = 0,$$

Solution

You can verify readily that the lines x - y + 4 = 0, x + y + 2 = 0, meet at (-3, 1), and writing x = -3 + x, y = 1 + y, you will find that the equation becomes

$$(x - y)\, dx - (x - y)\, dy = 0,$$

which is homogeneous. The substitution y = vx now gives

$$x(1 - v)\, dx - x\,(1 + v)\,(xdv + vdx) = 0$$

$$\text{or}\quad \frac{1 + v}{1 - 2v - v^2}\, dv = \frac{dx}{x}$$

Integrating, you have that

$$\frac{1}{2}\log(v^2 + 2v - 1) = \log\left(\frac{A}{X}\right)$$

and hence that

$$y^2 + 2xy - x^2 = B, \quad B = A^2$$

You may finally re-substitute X = x + 3, Y = y - 1 to obtain the general solution

$$(y - 1)^2 + 2(x + 3)(y - 1) - (x + 3)^2 = B.$$

Example 4.5

Solve the equation

$$\frac{dy}{dx} = \frac{2x + 2y + 3}{x + y - 1}$$

Solution

Since the lines 2x + 2y + 3 = 0, x + y - 1 = 0 are parallel, set z = x + y. Then

$$\frac{dz}{dx} - 1 = \frac{2z + 3}{z + 1}$$

$$\text{or } \frac{dz}{dx} = \frac{2z + 4}{z + 1}$$

From this, you can verify that

$$\left(\frac{z - 1}{z - 2}\right)dz = 2dx$$

$$or \left(1 - \frac{3}{z + 2}\right)dz = 2dx$$

Therefore z - 3 log (z +2) = 2x + A

and the general solution (y - x) + 3 log (x + y + 2) = A

Post-Test

1. Solve (c) and (e) in Example 4.1
2. solve the following equations
   (a) $(4y + x)dy - (y - 4x)dx = 0$
   (b) $(x + y)^2 \frac{dy}{dx} = x^2 - 2xy + 5y^2$
   (c) $y - x\frac{dy}{dx} + \sqrt{x^2 + y^2}$

(d) $(x^2 + 2xy)\frac{dy}{dx} + 2xy + y^2 + 3x^2 = 0$

given that $y = 2$ when $x = 1$.

3. Solve the following equations

(a) $(2x - 4y + 5)dy - (x - 2y + 3)\,dx = 0$

(b) $(x - 5y + 5)dx + (5x - y + 1)dy = 0$

(c) $\frac{dy}{dx} = \frac{3x - y + 4}{6x - 2y + 1}$

(d) $(x + y)^2 dy = (x + y + 2)^2\,dx$

LECTURE FIVE

Linear Equations

Introduction

If a differential equation can be rewritten in the form

$$\frac{dy}{dx} + py = Q \qquad \text{——} 5.1$$

Where P, Q are functions of x only, then the equation is said to be linear of first order, since dy/dx and y occur linearly. For example

$$\frac{dy}{dx} + (\sin x)\, y = x^2 - 1$$

is linear with $P = \sin x$, $Q = x^2 - 1$ and

$$\frac{1}{x^2}\frac{dy}{dx} - 4y = 7x^4$$

is seen to be linear after multiplying all through by x^2 to give:

$$\frac{dy}{dx} - 4x^2y = 7x^6$$

Example 5.1

Which of the following equations are linear. Recognise the types of the other equations:

(a) $x^4 \frac{dy}{dx} + (x^2 + 1)y = \tan x$

(b) $x^3 \frac{dy}{dx} - y^3 = x^2 y$

(c) $\frac{dy}{dx} - \frac{x}{1 + x^2} y = \frac{1}{x(1 + x^2)}$

(d) $\cos x \sin y \, dx + \sin x \cos y \, dy = 0$

(e) $\frac{dy}{dx} = 2x + y$

Solution

(a) Linear with $P = \frac{(x^2 + 1)}{(x^4)}$, $Q = x^{-4} \tan x$

(b) Homogeneous

(c) Linear $P = \frac{-x}{1 + x^2}$, $Q = \frac{1}{x(1 + x^2)}$

(d) Variable separable

(e) Linear: $P = \frac{1}{x}$, $Q = 2$; also homogeneous

Now that you can recognise linear first order equation let me introduce you to a method of solution.. In equation 5.1 the presence of the terms dy/dx and y suggests differentiation of a product involving y. To produce this product multiply 5.1 by a function $u = u(x)$, where u is to be determined later. This will give

$$u \frac{dy}{dx} + uPy = uQ \qquad \text{—— 5.2}$$

since $\frac{d}{dx}(uy) = \frac{u\,dy}{dx} + \frac{y\,du}{dx}$, you can rewrite 5.2 thus:

$$\frac{d}{dx}(uy) + uPy - \frac{y\,du}{dx} = uQ \qquad \text{—— 5.3}$$

This equation can be solved by direct integration if the term $(uPy - y\frac{du}{dx})$ vanished; that if

$$\frac{du}{dx} = Pu$$

which is an equation of the variable separable form. Separation of variables gives

$$u = e^{\int p\,dx} \qquad \text{—— 5.4}$$

with this choice of u 5.4, 5.3 reduces to

$$\frac{d}{dx}(uy) = uQ \qquad \text{—— 5.5}$$

and the general solution is

$$uy = \int uQdx + A \qquad \text{—— 5.6}$$

with u given by 5.4

To solve a linear first order equation you may proceed as follows:

(i) Write the equation in the standard form $\frac{dy}{dx} + Py = Q$

(ii) Evaluate $\int Pdx$ (no arbitrary constant is needed) and $u = e^{\int pdx}$

(iii) Multiply the standard form equation 5.1 by u and verify that if reduces to the form 5.5,

(iv) Integrate to obtain 5.6

Example 5.2

Solve the equation

$$x\frac{dy}{dx} - y = x + 1.$$

Solution

First, you reduce to standard form by dividing all through by x to obtain

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{x+1}{x},$$

so that $P = \frac{-1}{x}$, $Q = 1 + \frac{1}{x}$ Now

$$U = e^{\int Pdx} = e^{\log(1/x)} = 1/x$$

Therefore, on multiplying the last equation by 1/x, you have

$$\frac{d}{dx}\left(y\frac{1}{x}\right) = \frac{x+1}{x^2}$$

and by direct integration

$$\frac{y}{x} = \frac{1}{\log x - x} + A$$

Thus the general solution is

$$y = x\log x + Ax - 1$$

Note that the most crucial step in the solution of a linear first order equation is step (ii) above namely the evaluation of the integrating factor: $u = e^{\int pdx}$

The step (iii) can be omitted; its inclusion was to ensure that the correct expression for us is obtained.

In evaluating the integrating factor u, the integral. $\int pdx$ often involves the logarithm of a function. You will find, in this connection, the following identity useful

$$e^{\log_e f(x)} = \exp \log_e f(x) = f(x).$$

For example, if $p = 2/3 \cot 2x$, then

$\int Pdx = -1/3 \log_e \sin 2x = \log_e (\text{cosec } 2x)^{1/3}$

and $u = e^{\int Pdx} = (\text{cosec } 2x)^{1/3}$

Also if $P = \frac{x + 4}{x + 4}$, then

$\int pdx = \int (1 + 1/x + 3)dx = x + \log_e (x + 3)$

Hence $p = e^{\int Pdx} = e^x + \log_e (x + 3)$

$= e^x (x + 3)$

Example 5.3

Solve the equation

$$\frac{dy}{dx} + \left(\tan x - \frac{1}{x}\right)y$$

Solution

Here $P = \tan x - 1/x$ and

$\int Pdx = -\log \cos x - \log x$

$= \log \sec x - \log x$

$= \log \left(\frac{\sec x}{x}\right)$

Therefore $u = e^{\int Pdx} = \frac{\sec x}{x}$ and

$$\frac{\sec x}{x} y = \int \frac{\sec x}{x} \cdot x^2 \cos x + A$$

$$= \frac{1}{2} x^2 + A,$$

using formula 5.6. The general solution is

$$y = \cos x \left[\frac{1}{2} x^3 + Ax\right]$$

Example 5.4

Solve the equation

$$x(x^2 - 1) \frac{dy}{dx} + y = 2x^3.$$

Solution

Dividing through by $x(x^2 - 1)$, you will obtain

$$\frac{dy}{dx} + \frac{1}{x(x^2-1)}y = \frac{2x^2}{x^2-1},$$

So that $P = \dfrac{1}{x(x^2-1)}$. Now you can show by partial fractions that

$$\frac{1}{x(x^2-1)} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

and hence that

$$\int p\,dx = -\log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1)$$

$$= \log\left[\frac{(x^2-1)^{1/2}}{x}\right]$$

Therefore the integrating factor u is

$$u = \frac{(x^2-1)^{1/2}}{x}$$

and hence $y\dfrac{(x^2-1)^{1/2}}{x} = 2\int \dfrac{x}{(x^2-1)^{1/2}} + A$

$$= 2(x^2-1)^{1/2} + A$$

The general solution is

$$y = 2x + Ax(x^2-1)^{-1/2}$$

Post-Test

1. Obtain the integrating factor corresponding the following expression for

(a) $-\dfrac{(x+1)}{x}$ (b) $\dfrac{x^u}{1+x^5}$

(c) $\dfrac{1}{x(x^2+1)}$ (d) $\dfrac{3x}{(6x^2-1)^{1/2}}$

2. Solve the following equations:

(a) $x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = x^3$

(b) $2(x^2+x+1)\dfrac{dy}{dx} + (2x+1)y = 8x^2 + 1$

(c) $(\sin x \cos^3 x)\frac{dy}{dx} + (\cos 2x \cos^2 x)y = (\sin 2x)^{1/2}$

(d) $\text{Cos } x \frac{dy}{dx} - 3y\sin x = \cot x$

(e) $\sqrt{1 + x^2}\frac{dy}{dx} + y = 2x$

(f) $\frac{dy}{dx} + y \tan hx = 8e^{2x}$

(g) $y^1 + 2y \text{ cosec } 2x = 2 \cot^2 x \cos 2x$

(h) $(1 + x^2)^2 y^1 + (1 + x)(1 + x^2)y = 2x.$

Exact Equations

Introduction

Consider the function of two variables u(x,y). If you **hold** y **fixed, you may** differentiate with respect to x. The resulting expression is **called the partial derivative** of u(x,y) with respect to x and is denoted by $\frac{\delta u}{\delta x}$. **For example if** u(x,y) = x^2y^3, then

$\frac{\delta u}{\delta x} = 2xy^3$ and $\frac{\delta u}{\delta y} = 3x^2y^2$. In a similar way you can **define the partial** derivative of u with respect to y. The expression

$$\frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy$$

is called the Total Differential of u and is denoted by du.

Thus

$$du = \frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy. \qquad \text{—— 6.1}$$

For example the total differential of x^2y^3 is $2xy^3dx + 3x^2y^2\,dy$

Now when the general solution of a first order equation involves an arbitrary constant A explicitly, say u(x,y) = A —— 6.2

the operation of taking total differential eliminates A automatically, thus:

$$du(x, y) = 0 \qquad \text{—— 6.3}$$

or

$$\frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} = 0 \qquad \text{—— 6.4}$$

Reversing the situation, suppose you start with the differential equation

$$P(x,y)\, dx + Q(x,y)\, dy = 0 \qquad \text{—— 6.5}$$

If you can find a function u(x,y) such that

$$\frac{\delta u}{\delta x} = P(x,y), \ \frac{\delta u}{\delta y} = Q(x,y) \qquad \text{—— 6.6}$$

then 6.5 becomes du = 0, so that u(x,y) = A is the general solution of 5.5. In this case Pdx + Qdy is an **exact differential**, and 5.5 is called an exact differential equation.

To determine a criterion for a first order equation to be exact, recall that

$$\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x},$$

so that from 6.6

$$\frac{\delta p}{\delta y}(x,y) = \frac{\delta^2}{\delta x \delta y} = \frac{\delta Q}{\delta x}(x,y)$$

Therefore the condition for equation 6.5 to be exact is

$$\frac{\delta p}{\delta y} = \frac{\delta Q}{\delta x} \qquad \text{—— 6.7}$$

Example 6.1

Which of the following are exact?

(a) $2xydx + (x^2 + 1)dy = 0$

(b) $(x - y \cos x)dx - \sin x\, dy = 0$

(c) $y^2 dx + \frac{x}{y} dy = 0$

(d) $(3xt + 2y)dx + xdy = 0$

Solution

(a), (b) are exact, while (c) and (d) are not.

To obtain the general solution of 6.5 start from the first of the two relations in 6.6

$$\frac{\delta u}{\delta x}(x,y) = P(x,y)$$

Integrating partially with respect of x, you will obtain

$$u(x,y) = \int P(x,y)dx + \phi(y),$$

where the "constant of integration" $\phi(y)$ here is an arbitrary function of y which vanishes under the reverse process of differentiating partially with respect to x. Setting $\int P(x,y)dx = M(x,y)$, you have that

$$u(x,y) = M(x,y) + \phi(y) \qquad \text{—— 6.8}$$

But from 6.6

$$Q(x,y) = \frac{\delta u}{\delta y} = \frac{\delta M}{\delta y} + \phi^1(y) \qquad \text{—— 6.9}$$

and $\phi^1(y)$ can be determined from this, since Q and M are both given. Integration of 6.9 will give $\phi(y)$ together with an arbitrary constant of integration, and this combined with 6.8 will yield the general solution.

Example 6.2

You have seen that (a) of Example 6.1 is exact. Now solve the equation:

$$2xy\,dx + (x^2 + 1)dy = 0$$

Solution

Here $P = 2xy$, $Q = x^2 + 1$ and

(i) $\frac{\delta u}{\delta x} = p = 2xy$

(ii) $\frac{\delta u}{\delta y} = Q = x^2 + 1$ —— 6.10

Now integrate

$$\frac{\delta u}{\delta x} = 2xy$$

partially with respect to x to give $u(x,y) = x^2y + \phi(y)$ —— 6.11

To determine $\phi(y)$ differentiate 6.11 with respect to y and use 6.10 ii) above to obtain

$$x^2 + 1 = \frac{\delta u}{\delta y} = x^2 + \emptyset'(y).$$

Therefore $\phi(y) = y + A$ and from 6.11,

$$u(x,y) = y(x^2 + 1) + A.$$

Example 6.3

Solve the equation (b) Example 6.1.

Solution

Here $P = x - y\cos x$, $Q = -\sin x$ and

(i) $\frac{\delta u}{\delta x} = x - y\cos x$.

(ii) $\frac{\delta u}{\delta y} = -\sin x$ —— 6.12

Integrating 6.12 (ii) with respect to x: $u(x,y) = \frac{x^2}{2} - y\sin x + \varphi(y)$

and hence

$$-\sin x = \frac{\delta u}{\delta y} = -\sin x + \varphi^1(y).$$

Therefore $\varphi(y) = A$

and $u(x,y) = \frac{x^2}{2} - y\sin x + A$ is the general solution.

Equations Reducible to Exact Form

It is not in every case that the condition 6.7 for exactness can be met. In fact there are few equations which meet the condition. For example, you may verify that $(3x + 2y)dx + dy = 0$ —— 6.13

is not exact. However if you multiply the equation by x, the new equation which is of the form.

$$(3x^2 + 2xy)dx + x^2dy = 0$$

is exact, as you can verify. The factor x is called an **Integrating Factor.** More generally, given an equation

$$P(x,y)dx + Q(x,y)dy = 0,$$

which is not exact. If multiping by some function $\mu = \mu(x,y)$, the resulting equation

$$\mu P(x,y)dx + \mu Q(x,y)dy = 0 \qquad \text{—— 6.14}$$

becomes exact, then $\mu = \mu(x,y)$ is called an integrating factor. Finding an integrating factor is in general difficult. There is however one procedure which is sometimes successful. Assuming that 6.14 is exact, then from the condition of exactness you will see that the condition

$$\frac{\delta}{\delta y}(\mu p) = \frac{\delta}{\delta x}(\mu Q)$$

$$\text{or } \frac{1}{\mu}\left(p\frac{\delta\mu}{\delta y} - Q\frac{\delta\mu}{\delta x}\right) = -\left\{\frac{\delta p}{\delta y} - \frac{\delta Q}{\delta x}\right\} \qquad \text{—— 6.15}$$

must hold. If the integrating factor μ depends only on x, equation 6.15 becomes

$$\frac{1}{\mu}\frac{d\mu}{dx} = \left(\frac{\delta p}{\delta y} - \frac{\delta Q}{\delta x}\right)\Big/ Q = Qk \qquad \text{—— 6.16}$$

Since the left-hand side is a function of x only k must also be a function of x alone, and you can find μ by separating variables to obtain

$$\mu(x) = \exp\ [\ \int k(x)dx\ ]$$

A similar result holds if μ is a function of y only. In this case

$$k = \left(\frac{\delta p}{\delta y} - \frac{\delta Q}{\delta x}\right)\Big/ P \qquad \text{—— 6.17}$$

is also a function of y only and $\mu(y) = \exp\ [\ \int k(y)dy\ ]$.

Example 6.4

Show that e^x is an integrating factor of

$$(x + y)dx + dy = 0,$$

and hence obtain the general solution.

Solution

You will see here that P = x + y, Q = 1 and

$$\frac{\delta p}{\delta y} = 1 \neq \frac{\delta Q}{\delta x} = 0$$

so that the given equation is not exact. Multiplying all through by e^x, you have

$$e^x(x + y)dx + e^x dy = 0$$

and

$$\frac{\delta p}{\delta y} = \frac{\delta}{\delta y}[e^x(x + y)] = e^x = \frac{\delta Q}{\delta x} = \frac{\delta}{\delta x} e^x$$

Therefore e^x is an integrating factor. You may now integrate $\frac{\delta u}{\delta y} = e^x$

partially with respect to y to obtain $u = ye^x + \varphi(x)$.

But $e^x (x + y) = \frac{\delta u}{\delta x} = ye^x + \varphi^1(x)$

Therefore $\varphi^1(x) = xe^x$

and so $\varphi(x) = e^x(x - 1) + A$.

The general solution is

$$(x + y - 1)\, e^x = A$$

Example 6.5

Solve the equation $\qquad ydx + (y - x)dy = 0$

Solution

$P = y,\ Q = y - x$ and $\frac{\delta p}{\delta y} - \frac{\delta Q}{\delta x} = 2$

Therefore $K = \frac{-2}{y}$ and

$\mu(y) = y^{-2}$

Multiply the given equation by y^{-2} to obtain

$y^{-1}dx + \frac{(y-x)}{y^2}dy = 0$

Check that this equation is now exact. You may now integrate

$\frac{\delta u}{\delta x} = y^{-1}$

to obtain

$u = \frac{x}{y} + \varphi(y)$

Therefore

$\frac{y - x}{y^2} = \frac{\delta u}{\delta y} = \frac{-x}{y^2} + \varphi^1(y)$

and so

$\varphi(y) = \log y$

The general solution is

$\log y + \frac{x}{y} = A$

Post-Test

In each of the following exercises verify that the given equation is exact and find the general solution. Find a particular solution when an initial condition is given.

1. $[\, x\cos(x + y) + \sin(x + y)\,]\, dx + x\cos(x + y)dy = 0$

   $y(i) = \frac{\pi}{2} \quad 1$

2. $[\frac{\log(\log y)}{x} + \frac{2}{3}xy^3]\ dx + [\frac{\log x}{y\log y} + x^2y^2]\ dy = 0$
3. $\text{Cosh } 3x \cosh 3ydx + \sinh 3x \sinh 3y\, dy = 0$
4. $(x^2 + y^2)\, dx + 2xydy = 0$ $y(1) = 1$.
5. $\frac{(1 + y^2)\, ydx + (1 + x^2)\, xdy}{(1 + x^2 + y^2)^{3/2}} = 0$

6. $\left(\frac{1}{x} - \frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2+y^2} - \frac{1}{y}\right)dy = 0$

7. $(ax + hy)\, dx + (hx + by)\, dx = 0$

8. $2\left(\frac{x + a}{y + b}\right) dx - \left(\frac{x + a}{y + b}\right)^2 dy = 0$

9. Find the integrating factor for each of the following equations and obtain the general solution:

   (a) $(x^2 + y^2 + x)dx + ydy = 0$

   (b) $(x^2 + 2y)dx - xdy = 0$

   (c) $2y^2dx + (2x + 3xy)dy = 0$

10. Solve the equations:

   (a) $(x^2 - 2y^3 - 3xy)dx + 3x(y^2 + x)dy = 0$

   (b) $dx + \{ 1 + (x + y) \tan y \}\, dy = 0$

   (c) $(x^2 - y^2 + 1)dx + (x^2 - y^2 - 1)dy = 0$

   (d) $(7x^3 + 3x^2y + 4y)dx + (4x^3 + x + 5y)dy = 0$

LECTURE SEVEN

Some Special Equations

Introduction

You now have a number of techniques for solving first order equation. To conclude the discussion of first order equations the discussion of first order equations here are some special equations which can be reduced to the types that have been discussed.

The Bernoulli Equation

The equation $\frac{dy}{dx} + P(x)y = q(x)y^n$ —— 7.1

in which P, q are functions of x alone is associated with the name of James Bernoulli. Although a non-linear equation, it can be reduced to linear equation by a change of independent variable y. Indeed dividing all through by y^n gives

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = q(x) \qquad \text{—— 7.2}$$

and this suggests the substitution

$$v = y^{1-n} \text{ so that } v' = (1-n)y^{-n}\frac{dy}{dx}.$$

The equation 7.2 becomes

$$\frac{dv}{dx} + (1-n)\, p(x)v = (1-n)q(x) \qquad \text{—— 7.3}$$

which is the standard linear equation in the new independent variable v.

The Riccati Equation

This equation is of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

where p is not identically zero. It may be solved completely when one particular solution, say $y = y^1$ is known. This is achieved by the aid of the substitution

$$y = y^1 + \frac{1}{v}$$

which gives

$$\frac{dv}{dx} + (2py^1 + q)v + p = 0 \qquad \text{—— 7.4}$$

a linear equation in v

Higher Order Equations - Reduction of Order

The simplest case, where reduction of order is possible occurs when the dependent variable is absent from the equation, say, for example,

$$F(x, y', y'') = 0 \qquad \text{—— 7.5}$$

where $y'' = \frac{d^2y}{dx^2}$

The technique here is to replace y_1 by P and regard P as the new independent variable. This will reduce the equation to first order:

$$F(x, p, p') = 0$$

which hopefully you may solve using one of the techniques discussed before.

In particular, when x is absent as well as y (in 7.5) the equation can be written as

$$y'' = g(y')$$

$$\text{or } p' = g(p)$$

which is of the variable separable form. An equation of the type

$$F(x, y^{(n-1)}, y^{(n)}) = 0$$

can also be reduced to first order by the substitution $z = y^{(n-1)}$

Example 7.1

Solve the equation

$$\frac{dy}{dx} - \frac{y}{4x} = 3xy^5$$

Solution

Dividing by y^5 and setting $v = y^{-4}$, you have

$$v^1 + \frac{1}{x} v = -12x$$

with integrating factor x. Therefore $x v = -6x^2 + A$

$$\text{or } y = \sqrt[4]{\frac{x}{A - 6x^2}}$$

since $v = y^{-4}$

Example 7.2

Solve $2x^2 y^1 = (x - 1)(y^2 - x^2) + 2xy$

Solution

This is a Riccatti equation with obvious particular solutions $y = x$, $y = -x$. Substituting $y = x + \frac{1}{v}$

gives $2x^2(v_1\ v) = 1 - x$,

with the general solution

$v = (Axe^{-x} - 1)/2x$.

The general solution of the given equation is

$$y = x + \frac{2x}{Axe^{-x} - 1}$$

Second and Higher Order Equations with Constant Co-efficients

Introduction

The most general form of second order o.d.e. is $F(x, y, y', y'') = 0$, that is an equation involving x, y, y' and y''. There is no procedure for solving arbitrary equations of this form. However, one particular type occurs ever so often in applications arising from Science and Engineering.

This type is the simpliest of all second order o.d.equations and is of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{—— 8.1}$$

where **a, b, c are constants** and Q is a function of x only. Equation 8.1 is called a linear **second order equation** with constant co-efficients, the co-efficients being a, b, c. You **will, in this section**, learn some methods for solving such equations.

The solution of the o.d.e. 8.1 is closely related to that of the simpler one

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{—— 8.2}$$

Note that the difference between this equation and 8.1 is that Q = 0 in 8.2. The equation 8.2 is called a linear homogeneous constant co-efficient second order equation. It is also referred to as the **reduced equation** (of 8.1, since Q = 0).

The question you will ask is:

How do I solve this 'simple' equation? To answer this question, recall that the solution of the simplest first order equation

$$\frac{dy}{dx} + ky = 0 \text{ is } y = Ae^{-kx}$$

This form of solution is to be expected since the equation merely states that the derivative of y is a (negative) multiple of y itself, and the exponential function e^{kx} has such a property. Since the second derivative of an exponential function is also a multiple of itself, it is reasonable to expect that 8.2 might also have an exponential function as solution.

Now make $y = e^{mx}$, m a constant, as a trial solution of 8.2 since

$$\frac{dy}{dx} = me^{mx} \text{ and } \frac{d^2y}{dx^2} = m^2e^{mx}$$

you will see, on substituting into 8.2, that

$$am^2 + bm + c = 0, \qquad \text{—— 8.3}$$

which is a quadratic equation from which m is to be determined.

To fix ideas, consider the equation

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 2e^{-x} \qquad \text{—— 8.4}$$

with corresponds to 8.1 with a = 1, b = -3, c = 2 and Q = $2e_{-x}$

The reduced equation is

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0 \qquad \text{—— 8.5}$$

and making $y = e^{mx}$ a trial solution, so that $y_1 = me^{mx}$ $y^{II} = m^2e^{mx}$, gives

$$e^{mx}(m^2 - 3m + 2) = 0;$$

that is $m^2 - 3m + 2 = 0$

since $e^{mx} \neq 0$. This equation is called the **Auxillary or Characteristic equation**.

You may solve it to obtain

m = 1 or m = 2

You will then see that $y = e^x$ and $y = e^{2x}$ are the corresponding solutions of 8.5

You will note that two independent solutions have been obtained, each corresponding to a distinct root of the auxilliary equation. You may wish to verify by direct substitution that $y = Ae^x$, $y = Be^{2x}$ and $y = Ae^x + Be_{2x}$ are solutions of the reduced equation, A and B being arbitrary constants. The solution

$$y = Ae^x + Be^{2x}$$

is the general solution of 8.5 since it contains two arbitrary constants; it is also referred to as the **complimentary function** of 8.4.

Returning now to the equation 8.4, you will readily check that the complimentary function is not its solution; an extra term needs to be added to balance, the $2e^{-x}$ on the right hand side. You may obtain this extra term by making a **trial solution** of $y = \alpha e^{-x}$. Indeed substitution in 8.4 will give

$$(\alpha + 3\alpha + 2\alpha)e^{-x} = 2e^{-x},$$

so that $\alpha = \frac{1}{3}$. Therefore the complete solution of 8.4 is

$$y = Ae^{x} + Be^{2x} + \frac{1}{3}e^{-x}$$

since this contains two arbitrary constants it is the **general solution** of 8.4. The extra trial solution $\frac{1}{3}e^{-x}$ is referred to as a **particular integral** of 8.4 since it contains no arbitrary constant (i.e. the constant α having been assigned the value 1/3). Thus you now see that the general solution of 8.4 is the sum of the complimentary function and a particular integral.

Complimentary Function

You will recall that substitution of the trial solution $y = e^{mx}$ into the reduced equation 8.2

$$ay'' + by' + cy = 0$$

gave rise to the auxilliary equation 8.3,

$$am^2 + bm + c = 0$$

From what you know about quadratic equation, the two roots m_1, m_2 are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and the following three cases are possible:

(i) real unequal roots, if $b^2 > 4ac$;

(ii) real but equal roots, if $b^2 = 4ac$;

(iii) complex conjugate roots, if $b^2 < 4ac$.

The two independent solutions giving the complimentary function are determined by m_1 and m_2.

You need to know the form of the complimentary function corresponding to each of the three cases.

Case 1

If the roots m_1, m_2 are distinct real numbers, then the complimentary function is

$$y = Ae^{m_1 x} + Be^{m_2 x} \qquad \text{—— 8.5}$$

Example 8.1

Solve

$$y'' + 3y' - 10y = 0$$

Observe first the close similarity between the reduced equation 8.2 and the auxilliary equation 8.3. Indeed replacing y'' by m^2 and y' by m, you obtain 8.3 from 8.2. Therefore, in this case, the auxilliary equation is

$$m^2 + 3m - 10 = 0$$

or $(m + 5)(m - 2) = 0$

Therefore $m_1 = -5$ and $m_2 = 2$

The general solution is

$$y = Ae^{-5x} + Be^{2x}$$

Case II

Here $m_1 = m_2 = n$ say, and as in the previous case, you might think that the complimentary function is $y = Ae^{nx} + Be^{nx} = Ce^{nx}$ where $C = A + B$. This means that the solution contains only one arbitrary constant instead of two and hence it is not the complete complimentary function.

To complete the complimentary function make a trial solution of $y = e^{nx}v$, where v is a function of x to be determined. Observe here that $n = \frac{-b}{2a}$ since $an^2 + bn + c = 0$ and $b^2 = 4ac$. If you now substitute

$y' = e^{nx}(v' + nv)$, $y'' = e^{nx}(v'' + 2nv' + n^2v)$ into 8.1, you will see, after some rearrangement of terms that,

$$e^{nx}[a(v'' + 2nv' + n^2v) + b(v' + nv) + cv] = 0$$

or

$$av'' + (2an + b)v' + (an^2 + bn + c)v = 0$$

since the co-efficients of v' and v reduce to zero, this gives

$$v'' = 0$$

and direct integration yields

$$v = Ax + B,$$

where A, B arbitrary constants. Thus the complete primitive is

$$y = (Ax + B)e^{nx} \qquad \text{—— 8.6}$$

You may verify by direct differentiation that 8.6 is indeed a solution of 8.3.

Example 8.2

Solve $y'' - 2y' + y = 0$

Solution

Here the auxilliary equation $m^2 - 2m + 1 = 0$ has $m = 1$ as a repeated root. Therefore the general solution is

$$y = (A + Bx)e^x.$$

Case III

Here $b^2 < 4ac$ and the roots m_1, m_2 of 8.3 are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = P + iq$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = P - iq$$

say, where $p = \frac{-b}{2a}$, $q = \frac{\sqrt{4ac - b^2}}{2a}$, $i^2 = -1$.

The complimentary function is

$$y = A_1e^{(p+iq)x} + B_1e^{(P-iq)x}$$

where A_1, B_1 are arbitrary constants. Using the fact that $e^{iz} = \cos z + i \sin z$, this expression you can rewrite as

$$y = e^{px}[A_1(\cos qx + i \sin qx) + B_1 (\cos qx - i \sin qx)]$$

$$= e^{px}[A \cos qx + B \sin qx]$$

where $A = A_1 + B_1$, $B = i(A_1 - B_1)$. Observe that the constant A, B need not be complex. Indeed, by direct substitution you can verify that, $y = e^{px} \cos qx$, $y = e^{px} \sin qx$ are solutions of 8.1. Therefore for arbitrarily real constants A and B,

$$y = Ae^{px} \cos qx + Be^{px} \sin qx \qquad \text{—— 8.7}$$

is the general solution of 8.3 with

$$A = C \sin \theta, B = \cos \theta, \text{ so that } C = \sqrt{A^2 + B^2}$$

the last expression can be rewritten thus:

$$y = Ce^{px} \sin (qx + \theta), \qquad \text{—— 8.8}$$

which is another form for the general solution of 8.3. Yet another form is

$$y = Ce^{px} \cos (qx - \phi) \qquad \text{—— 8.9}$$

with $A = C \cos \phi$ and $B = C \sin \phi$.

Example 8.3

Solve $y'' + y' + y = 0$.

Solution

Here the auxilliary equation is $m^2 + m + 1 = 0$ with roots

$m_1 = (-1 + i\sqrt{3})/2, \quad m_2 = (-1 - i\sqrt{3})/2.$

Therefore the general solution is

$$y = e \quad [A \cos \frac{\sqrt{3}}{2} x \quad + B \sin \frac{\sqrt{3}}{2} x].$$

SUMMARY

Observe that we have talked of two independent solutions of a second order linear o.d.e. without defining the term independent. Two solutions of a second order linear o.d.e. are said to be linearly independent if one cannot be written as a multiple of the other. Thus, in example 8.1, for instance, e^{-5x}, e^{2x} are two linearly independent solutions since there does not exist a constant $\alpha \neq 0$ such that $e^{-5x} = \alpha e^{2x}$ for all x.

There is, however some other test for linear independence. Let $y_1(x)$, $y_2(x)$ be any two solutions of 8.2. The **Wronskian** of y_1, y_2, denoted by $W(y_1, y_2)$ is defined to be

$$W(y_1, y_2) = y_1(x)y_2^1(x) - y_2(x)y_1^1(x)$$

It turns out that the following result is true:

The solutions $y_1(x)$, $y_2(x)$ of equation 8.3 are linearly independent on $[x_0, x_1]$ if and only if $W(y_1, y_2) \neq 0$.

Finally note that the general solution of 8.2 (that is, the complimentary function) is a linear combination of two linearly independent solutions of 8.2.

Post-Test

Solve each of the following equations:

1. $2y'' + 3y' - 2y = 0$
2. $y'' - 2y' + 2y = 0$
3. $y'' - 9y = 0$
4. $y'' + 6y' + 9y = 0$
5. $y'' + 9y = 0$

LECTURE NINE

Particular Integrals

Introduction

In the previous section you went through a number of methods of finding the complimentary function, depending on the form of the auxilliary equation. Since the general solution of 8.1 is the sum of the complimentary function and a particular solution, here you will learn how to find the particular integral when Q is of the following forms which occur most frequently in applications from Science and Engineering:

(1) $P_n(x)$

(2) $P_n(x)e^{ax}$

(3) $P_n(x)e^{ax}\sin bx$ or $P_n(x)e^{ax}\cos bx$

where $P_n(x)$ is a polynomial in x of degree n. The technique to be used is essentially the same as that employed for equation 8.4; that is the trial solution for the particular integral, P.I. will be assumed to have the same basic form as Q(x). This method is best illustrated by a number of examples.

Case 1

Consider $y'' - 4y = x^2$ —— 9.1

Since $Q(x) = x^2$ is a polynominal of degree 2, the suggested form of trial solution for the P.I. is

$$y_p(x) = a + bx + cx^2,$$

where a, b, c are to be determined in such a way that $y_p(x)$ is indeed a P.I. You will observe that $y_p'' = 2c$, so that substituting $y_p(x)$ into 9.1 gives

$$2c - 4(a + bx + cx^2) = x^2$$

Equating the co-efficients of powers of x, starting with the lowest power:

co-efficient of x^0 $\quad 2c - 4a = 0$

co-efficient of x $\quad -4b = 0$

co-efficient of x^2 $\quad -4c = 1$

Therefore $c = -\frac{1}{4}$, $b = 0$ and $a = -\frac{1}{8}$

The P. I. is

$$y_p(x) = -\frac{1}{8} - \frac{x^2}{4}.$$

The complimentry function, C.F., of 9.1 is

$$y_h(x) = Ae^{2x} + Be^{-2x}$$

and $y = Ae^{2x} + Be^{-2x} - \frac{1}{8} - \frac{x^2}{4}$

is the general solution.

Case 2

Consider $y'' + 4y = xe^x$ —— 9.2

$Q(x) = xe^x$ is of the form 2 with $p_n(x)$ a polynomial of degree one. You may therefore make a trial solution for the P.I. of the form

$$y_p(x) = e^x(a + bx)$$

where a, b are constants to be determined so that $y_p(x)$ is indeed a P.I. for 9.2. You can readily show that

$y'_p = e^x(a + b + bx)$, $y'' = e^x[a + 2b + bx]$, and on substituting in 9.2 that

$$e^x[a + 2b + bx] + 4e^x[a + bx] = xe^x.$$

Dividing by e^x and equating coefficients now give

$$5a + 2b = 0, 5b = 1$$

Thus $b = \frac{1}{5}$, $a = \frac{-2}{25}$

and a P.I. is

$$y_p(x) = \frac{e^x}{25}[5x - 2]$$

Since $Q(x) = x^2$ is a polynominal of degree 2, the suggested form of trial solution for the P.I. is

$$y_p(x) = a + bx + cx^2,$$

where a, b, c are to be determined in such a way that $y_p(x)$ is indeed a P.I. You will observe that $y_p'' = 2c$, so that substituting $y_p(x)$ into 9.1 gives

$$2c - 4(a + bx + cx^2) = x^2$$

Equating the co-efficients of powers of x, starting with the lowest power:

co-efficient of x^0 $\quad 2c - 4a = 0$

co-efficient of x $\quad -4b = 0$

co-efficient of x^2 $\quad -4c = 1$

Therefore $c = -\frac{1}{4}$, $b = 0$ and $a = -\frac{1}{8}$

The P. I. is

$$y_p(x) = -\frac{1}{8} - \frac{x^2}{4}.$$

The complimentry function, C.F., of 9.1 is

$$y_h(x) = Ae^{2x} + Be^{-2x}$$

and $y = Ae^{2x} + Be^{-2x} - \frac{1}{8} - \frac{x^2}{4}$

is the general solution.

Case 2

Consider $y'' + 4y = xe^x$ —— 9.2

$Q(x) = xe^x$ is of the form 2 with $p_n(x)$ a polynomial of degree one. You may therefore make a trial solution for the P.I. of the form

$$y_p(x) = e^x(a + bx)$$

where a, b are constants to be determined so that $y_p(x)$ is indeed a P.I. for 9.2. You can readily show that

$y'_p = e^x(a + b + bx)$, $y'' = e^x[a + 2b + bx]$, and on substituting in 9.2 that

$$e^x[a + 2b + bx] + 4e^x[a + bx] = xe^x.$$

Dividing by e^x and equating coefficients now give

$$5a + 2b = 0, \; 5b = 1$$

Thus $b = \frac{1}{5}$, $a = \frac{-2}{25}$

and a P.I. is

$$y_p(x) = \frac{e^x}{25}[5x - 2]$$

since $Q = 3 \cos 2x = Re(3e^{i2x})$, the trial solution for the P.I. is

$$y = \alpha e^{i2x}$$

Substituting in $y'' + 3y' + 2y = 3e^{i2x}$ —— 9.5

gives

$$\alpha[(i2)^2 + 3i2 + 2] e^{i2x} = 3e^{i2x}$$

and so

$$\alpha = \frac{3}{-2 + 6i}.$$

Therefore the P. I. for 9.5 is

$$\frac{3e^{i2x}}{-2 + 6i}$$

and, hence, the P.I. for 9.4 is

$$y_p = Re\left\{ \frac{3}{-2 + 6i} \right\} e^{i2x}$$

If you rationalise in the usual way, you will obtain

$$y_p = \frac{3}{20} (3 \sin 2x - \cos 2x).$$

The general solution of 9.4 is

$$y = Ae^{-x} + Be^{-2x} + \frac{3}{20} (3 \sin 2x - \cos 2x).$$

The advantage of this alternative method is two-fold: (i) only one parameter (constant) has to be determined instead of two in case 3, (ii) the differentiation and substitution are more compact and easier to carry out.

As a further example, consider

$$y'' + y' - 2y = 4e^{-x} \sin 3x \qquad —— 9.6$$

You can verify readily that the complimentary function is

$$y_h = Ae^{x} + Be^{-2x}$$

Now, you may rewrite the right hand side of 9.6 as

$$Q = 4e^{-x} \sin 3x = 1m[4e^{(-1 + i3)x}]$$

and hence try $y = \alpha e^{(-1 + i3)x}$ for P.I. of

$$y'' + y' - 2y = 4e^{(-1 + i3)x}$$

substitution gives

$$\alpha[(-1 + i3)^2 + (-1 + i3) - 2]e^{(-1 + i3)x} = 4e^{(-1 + i3)x}$$

and so,

$$\alpha = \frac{-4}{11 + i3}.$$

Therefore P.I. is

$$y_p = \text{Im}\left[\frac{-4}{11+i3} e^{(-1+i3)x}\right]$$

$$= \text{Im}\left[\frac{-2}{65} e^{-x}(\cos 3x + i \sin 3x)(11 - i3)\right]$$

On rationalizing, you readily have that

$$y_p = \frac{2}{65} e^{-x}(3 \cos 3x - 11 \sin 3x).$$

Principle of Superposition

If $Q(x) = Q_1(x) + Q_2(x)$ you may separately solve the equations

$ay'' + by' + cy = Q_1(x)$ with particular solution $y_{P_1}(x)$, and

$ay'' + by' + cy = Q_2(x)$ with particular solution $y_{P_2}(x)$.

The particular solution to

$ay'' + by' + cy = Q(x) = Q_1(x) + Q_2(x)$ is given by

$y_P(x) = y_{P_1}(x) + y_{P_2}(x)$.

consider the equation

$$y'' - 4y = x^2 + e^{-x} \qquad \text{—— 9.7}$$

using the result obtained under Case I a P.I. for $y'' - 4y = x^2$ is

$y_1(x) = -\frac{1}{4} - \frac{1}{8} x^2$. You can venvy readily thad a P.I. for

$y'' - 4y = e^{-x}$ is $y_2(x) = -\frac{1}{2} e^{-x}$.

Therefore, by the principle of superposition, a P.I. of 9.7 is

$$y_P(x) = -\frac{1}{4} - \frac{1}{8} x^2 - \frac{1}{2} e^{-x}.$$

Cases of Failure

You will recall that the form of the trial solution of the P.I. is determined by the form of Q (or of its components Q_1 and Q_2. In some cases; the forms of the function arising in the trial solution appear in the complimentary function, and the methods for determining the P.I. earlier discussed fail. For example, consider

$$y'' - 4y = 5e^{2x} \qquad \text{—— 9.8}$$

The C.F. is $Ae^{2x} + Be^{-2x}$ which contains the function e^{2x} appearing in Q. Indeed if you attempted a trial P.I. of $y = \alpha e^{2x}$, you will obtain

$$\alpha(4 - 4)e^{2x} = 5e^{2x}$$

showing that is indeterminate.

This is due to the fact that e^{2x} is a solution of the reduced equation $y'' - 4y = 0$. Such cases are referred to as cases of failure.

You will recall that when this repetition of functions ocurred in Case II, the remedy was to multiply the recuring one by x. A similar modification is used in the cases of failure, and the final trial solution for the P.I. of 9.8 is modified to $y = \alpha xe^{2x}$, on multiplying by x. Now substituting in 9.8 gives

$4\alpha e^{2x} + 4\alpha xe^{2x} - 4\alpha xe^{2x} = 5e^{2x}$ and so $\alpha = \frac{5}{4}$. The P.I. is $y_p(x) = \frac{5}{4}xe^{2x}$ and the general solution of 9.8 is

$$y = Ae^{2x} + Be^{-2x} + \frac{5}{4}e^{2x}.$$

In general if any term of $y_p(x)$ is a solution of 9.2 then multiply the appropriate function $y_p(x)$ by x^k, where k is the smallest integer such that no term in $x^k y_p(x)$ is a solution of 9.2.

Initial-Value Problem

Introduction

You know that the general solution of linear second order o.d. equation involves two arbitrary constants. In physical problems, solutions are usually desired that satisfy certain specified conditions. These conditions provide information from which values can be assigned to the arbitrary constants. This type of solution satisfying certain given conditions is called a 'Particular Solution' and the conditions satisfied are either **Initial Conditions** or **Boundary Conditions**. Naturally two boundary or initial conditions will be required since the equations considered here are of order 2.

Example 10.1

Consider the equation $y'' - 6y' + 9y = e^{2x}$ —— 10.1

subject to the conditions

$y(0) = 0, \; y'(0) = 1$ —— 10.2

A condition of the form 10.2 in which the value of the solution as well as its derivative is specified at a point ($x = 0$ in this case) is called **Initial Conditions**. The problem of solving 10.1 subject to 10.2 is called an **Initial Value** problem.

To obtain the desired particular solution, observe that the complimentary function of 10.1 is

$$y_h(x) = (A + Bx)e^{3x},$$

where A, B are arbitrary constants. Furthermore you can readily verify that a P.I. of 10.1 is

$$y_p(x) = e^{2x},$$

so that

$$y(x) = (A + Bx)\, e^{3x} + e^{2x} \qquad \text{—— 10.3}$$

is the general solution. Differentiating this expression you also have that

$$y'(x) = [(3A + B) + 3Bx]e^{3x} + 2e^{2x}$$

using 10.2, it is clear that

$$y(0) = A + 1 = 0$$

$$y'(0) = 3A + B + 2 = 1$$

and hence that $A = -1,\ B = 2$.

The desired Particular Solution is obtained by substituting these values of A and B in 10.3.

The Particular Solution is

$$y = (2x - 1)e^{3x} + e^{2x}$$

Now consider the equation

$$y'' + y = \sin 2x \qquad \text{—— 10.4}$$

subject to the condition

$$y(0) = 1,\ y(\tfrac{\pi}{2}) = -1 \qquad \text{—— 10.5}$$

A set of conditions, such as 10.5 in which the values of the solution is specified at two different points is called Boundary Conditions and the problem of solving 10.4 subject to 10.5 is called a boundary value problem.

You can show readily that

$$y_h(x) = A \sin x + B \cos x$$

is the complimentary function and that

$$y_p(x) = -\frac{1}{3} \sin 2x$$

is a P.I. of 10.4, where A, B are arbitrary constants. The general solution is

$$y = A \sin x + B \cos x - \frac{1}{3} \sin 2x.$$

Using 10.5, you have that

$y(0) = B = 1$

$y(-\frac{\pi}{2}) = A = -1.$

and hence the desired particular solution is

$y = \cos x - \sin x - \frac{1}{3} \sin 2x$

Post-Test

1. Find the general solution of each of the following equations. When initial or boundary conditions are given, find the particular solution:

   (a) $y'' + y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

   (b) $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 2$

   (c) $y'' - 5y' = 0$

   (d) $2y'' + 5y' - 3y = 0$

   (e) $y'' - 3y' + 2y = 0$

   (f) $y'' + 2\pi y' + \pi^2 y = 0, y(1) = 1, y'(1) = \frac{1}{\pi}$

2. Solve the following equations

   (a) $2y'' + 3y' - 2y = 0$

   (b) $y'' - 3y' + 2y = 2e^{-4x}$

   (c) $2y'' + 5y' - 3y = 2x^2 - x + 1$

   (d) $y'' + 2y' + y = x + \sin x, \quad y(0) = 1, y'(0) = 1$

   (e) $y'' + 2y' + y = x \cos x, \quad y(0) = y(\frac{\pi}{2}) =$

   (f) $y'' + 8y' + 17y = 2e^{-3x}$

   (g) $2y'' + y' - y = 3e^{-x} \sin x$

   (h) $y'' - 2y' = 5x e^{-x}$

3. Solve the following equations:

   (a) $y'' - y' - 2y = 5e^{2x}$

   (b) $y'' + y = 3 \sin x + x$

   (c) $y'' + 4y' = 2x$

   (d) $y'' - 4y' + 5y = 2e^{2x} \sin x$

   (e) $y'' + y' - 2y = 5e^{-x} \sin 2x, \quad y(0) = 1, y'(0) = 0$

   (f) $y'' + 8y' + 17y = 2e^{-3x}, \quad y(0) = 2, \quad y(\frac{\pi}{2}) = 0$

   (g) $y'' + 9y = 6 \cos 3x, y(0) = \frac{\pi}{4}, \quad y(\frac{\pi}{6}) = \frac{\pi}{3}$

   (h) $y'' - 2y' + 5y = (e^{-x} + e^{x}) \sin 2x$

Equations with Variable Co-efficients

Introduction

You now know how to solve second order o. d. equations when the co-efficients are constants. However, the situation is different for equations of the form

$$y'' + a(x)y' + b(x)y = Q(x)$$

where the co-efficients a, b are functions of x. In general there are no known general method for solving such equations. There are, however, certain special cases in which such equation can be solved. Two of such cases will be treated here.

Using One Solution to find another

Consider the reduced equation $y'' + a(x)y' + b(x)y = 0$ —— 11.1

where a, b are functions of x. It is sometimes possible to find one solution by inspecting the equation or by trial and error. When such a solution has been found, a standard procedure exists for finding a second linearly independent solution.

Assume that y_1 is a non-zero solution of 11.1 and seeks another independent solution y_2. If such a solution can be found, then a relationship of the form

$$\frac{y_2}{y_1} = V(x)$$

must exist such that v is not a constant, and $y_2 = vy_1$ must satisfy 11.1. That is

$(vy_1)'' + a(x)(vy_1)' + b(x)(vy_1) = 0,$ —— 11.2

or after differentiations, factoring and dropping dependence of a, b on x,

$v(y_1'' + ay_1' + by_1) + v'(2y_1' + ay_1) + v''y_1 = 0,$ —— 11.3

The term in parentheses in 11.3 is zero since y_1 is a solution of 11.2.

Therefore

$v''y_1 + v'(2y'_1 + ay_1) = 0$

or

$\frac{v''}{v'} = -\frac{2y'_1}{y_1} - a$ —— 11.4

on dividing all through by $v'y_1$.

Integrating, you have $\log v' = -2 \log y_1 - \int a(x)dx$,

and so

$v_1 = \frac{1}{y_1^2} e^{-\int a(x)dx}.$

Since the exponential is non-zero and v is non-constant, a second integration gives

$y_2 = vy_1 = y_1(x) \int \frac{e^{-\int a(x)dx}}{y_1^2(x)} dx$ —— 11.5

You can now verify that y_2 is a solution of 11.1. Indeed this folllows from the fact that v is a solution of 11.4 and vy_1 $(= y_2)$ is a solution of 11.2.

The formula 11.5 gives a second solution y_2 when a first y_1 is known.

Example 11.1

Solve $y'' - 2y + y = 0$

given that $y_1(x) = e^x$ is a solution.

Here $a(x) = -2$ and $e^{-\int a(x)dx} = e^{2x}$.

Therefore

$y_2 = e \int \left(\frac{e^{2x}}{e^{2x}}\right) dx = xe^x,$

and a second independent solution is $y_2(x) = xe^x$ The general solution is

$y = (A + Bx)e^x,$

where A, B are arbitrary constants.

Example 11.2

Solve $x^2y'' + xy' - 4y = 0$

given that $y_1 = x^2$ is a solution.

First rewrite the equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0,$$

so that $\int a(x)dx = \log x$ and $e^{-\int a(x)dx} = \frac{1}{x}$

From 11.5 $y_2 = x^2 \int x^{-5}dx = (-\frac{1}{4}x^{-4})x^2$

$$= -\frac{1}{4}x^{-2}$$

The general solution is $y = Ax^2 + Bx^{-2}$

The Euler Equation

This is of the form

$$x^2y'' + axy' + by = Q(x) \qquad \text{—— 11.6}$$

where a, b are constants. This equation can be reduced to one with constant coefficients by the substitution $x = e^t$. Indeed if you use dot to denote differentiation with respect to t, then

$$y' = \frac{dy}{dx} = \dot{y}\frac{dt}{dx} = \frac{1}{x}\dot{y}$$

and $y'' = \frac{1}{x^2}\ddot{y} - \frac{1}{x^2}\dot{y}$

Therefore 11.6 becomes

$$\ddot{y} + (a - 1)\dot{y} + by = Q(e^t)$$

which is of constant coefficient, since a, b are constant.

Example 11.3

Solve $x^2y'' - 2y' + 2y = x^3 \log x$ —— 11.7

Solution

Using the substitution $x = e^t$, the given equation becomes

$$\ddot{y} - 3\dot{y} + 2y = te^{3t}$$

The complementary function is

$$y_h(t) = Ae^t + Be^{2t}$$

and a P. I. is $y_p(t) = (t - 3)\, e^{3t}$

Hence, the general solution of the transformed equation is

$$y = Ae^{t} + Be^{2t} + (t - 3)e^{3t}$$

Using the transformation $x = e^t$, the general solution of 11.6 is given by

$$y = Ax + Bx^2 + x^3 (\log x - 3).$$

Example 11.4

Solve the equation

$$(x + 1)^2 y'' + (x + 1)y' - y = x \qquad \text{—— 11.8}$$

The substitution $x + 1 = e^t$ gives

$$y' = y \frac{dt}{dx} = \frac{1}{x + 1} y$$

$$y'' = \frac{1}{(x + 1)^2} y \quad \frac{1}{(x + 1)^2} y$$

and, the given equation becomes

$$y - y = e^t - 1$$

The complimentary function is

$$y_h^{(t)} = Ae^{-t} + Be^{t}$$

and a P. I. is $y_p(t) = \frac{1}{2} te^t + 1.$

Therefore the general solution of the reduced equation is

$$y = Ae^{-t} + Be^{t} + \frac{1}{2} te^{t} + 1$$

The general solution of 11.8 is now given by

$y = A(x + 1)^{-1} + B(x + 1) + \frac{1}{2}(x + 1) \log (x + 1) + 1$, **with A, B arbitrary** constants.

Post-Test

1. In each of the equations (a) - (d), one solution $y_1(x)$ is given. Verify that $y_1(x)$ is indeed a solution and find a second independent solution.
   (a) $(1 - x^2)y'' - 2xy' + 2y = 0$ $(|x| < 1)$ $y_1(x) = x$
   (b) $x^2y'' - xy' + y = 0$ $(x > 0)$ $y_1(x) = x$
   (c) $y'' - \left(\frac{2x}{1 - x^2}\right)y^1 + \left(\frac{6}{1 - x^2}\right) y = 0$, $y_1(x) = \frac{3x^2 - 1}{2}$, $|x| < 1$
   (d) $y'' + \left(\frac{3}{x}\right) y' = 0$, $y_1(x) = 1$

2. The Bessel equation is

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

For $p = 1/2$ verify that $y_1(x) = (\sin x)/\sqrt{x}$ is a solution for $x > 0$. Find a second independent solution.

3. Solve the following equations

(a) $x^2y'' + 4xy' + 2y = \log(1 + x)$

(b) $2(x + 1)^2y'' - (x + 1)y' + y = x$

(c) $x^{a+1}y'' - (2a - 1)x^a y' + a^2x^{a-1}y = 1$

4. Transform the following equation by the substitutions indicated, and hence solve the equation:

(a) $y'' \cos x + y' \sin x = y \cos^2 x$; $t = \sin x$

(b) $(1 + x^2)\, y'' + xy' = 4y$; $x = \sin ht$

(c) $(1 - x^2)\, y'' - xy' = 0$; $x = \sin t$.

Higher Order Equations

Introduction

In the preceeding section you learned how to solve some classes of equations of second order, namely constant co-efficients equations and some simple cases of variable co-efficient equations. In this section the techniques employed will be shown to be applicable to equations of order higher than two. Consider the n^{th} order linear differential equation.

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a(x)y' + a_n(x)y = Q(x) \qquad \text{—— 12.1}$$

and the reduced homogeneous equation

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a(x)y' + a_n(x)y = 0 \qquad \text{—— 12.2}$$

where the co-efficients $a(x)$, $i = 1,2, \ldots n$, are functions of x only and Q(x) also depend only on x. As in the second order case if you can find independent solutions y_i, y_2, ... y_n of 12.1 and a particular integral y_p of 12.2, then

$$y(x) = A_1y_1(x) + A_2y_2(x) + \ldots + Any_n(x) + y_p(x)$$

is the general solution of 12.1. Here A_1, A_2, ... An are n arbitrary constants. As in the second order equation, there is no general procedure for obtaining the independent solutions of 12.2 and a particular integral of 12.1.

However when the coefficients a_i i = 1, 2, ...n are all constants, the techniques employed for the corresponding second order equation carry over.

Consider the linear (real) constant coefficients n^{th} order homogeneous equation

$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + ... + a_{n-1}y_1 + a_ny = 0$ —— 12.3

where a_i, i = 1, 2, ...n are constants.

As in the second order case, make a trial solution of $y = e^{\lambda x}$. Substituting in 12.3 **and noting that k^{th} derivative of $e^{\lambda x}$ is** $\lambda^k e^{\lambda x}$, you will obtain

$$e^{\lambda x}(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n) = 0$$

so that $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0$ —— 12.4

since $e^{\lambda x} \neq 0$. The polynomial 12.4 is called the **auxilliary equation** for the homogeneous equation 12.3 and it is evident that if λ is a root of 12.4, then $e^{\lambda x}$ is a solution of 12.3.

As you must have known (or will learn) a polynomial $p(\lambda)$ of degree n can be factored in the form

$$p(\lambda) = (\lambda - \lambda_1)m_1(\lambda-\lambda_2)m_2...(\lambda-\lambda_k)m_k$$

where mi is the multiplicity (the number of repetition) of the root λ ;

and $m_1 + m_2 + ... m_k = n$. If $m_i = 1$ for a given j, then the root λ_j; is called a **simple root** of 12.4 since all the coefficients a_i are real, any complex roots of 12.4 will appear in conjugate; that is if $\lambda = \alpha + i\beta$ is a root of 12.4, so also is $\lambda = \alpha - i\beta$.

You will see readily that the nature of the solutions of 12.3, corresponding to different types of root, is as follows

(a) if $\lambda_1, \lambda_2, ..\lambda_p$ are p real simple roots of 12.4, then

$e^{\lambda_1 x}, e^{\lambda_2 x}, ... e^{\lambda_p x}$ are p independent solutions of 12.3.

(b) if λ_j is a root of 12.4 of multiplicity $m_j >$ I, then

$e^{\lambda_j x}, xe^{\lambda_j x}, x^2e^{\lambda_j x}, ... x^{(mj-1)} e^{\lambda_j x}$ are m_j independent solutions of 12.3.

(c) If $\lambda_j = \alpha + i\beta, \bar{\lambda}_j = \alpha - i\beta$ are a pair of simple complex conjugate roots of 12.4, then

$e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$ are two independent solutions of 12.3.

(d) If $\lambda_j = \alpha + i\beta$, $\bar{\lambda}_j = \alpha - i\beta$ are a pair of complex conjugate roots of multiplicity $m_j >$ I, then

$e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$, $xe^{\alpha x} \cos \beta x$, $xe^{\alpha x} \sin \beta x$,

$x^2e^{\alpha x} \cos \beta x$, $x^2e^{\alpha x} \sin \beta x$ .... $x^{mj-1}e^{\alpha x} \cos \beta x$,

$x^{mj-1} e^{\alpha x} \sin \beta x$ are $2m_j$ independent solutions of 12.3.

These results will be illustrated by a number of examples; their proofs are left as exercises. It should be observed here that finding the roots of polynominal of degree greater than 2 is generally a tedious task. The fact that the examples that follow have easily calculable roots should not deceive you. In the case of most higher order polynomials the roots can only be approximated, and methods for performing such approximations are dealt with in numerical analysis.

Example 12.1

Solve the equation

$$y_{iii} - 3y_i + 2y = 0 \qquad \text{—— 12.5}$$

Solution

Here the auxilliary equation is

$$\lambda^3 - 3\lambda + 2 = 0$$

By inspection you will find that $\lambda = 1$ is a root. What is more

$$\lambda^3 - 3\lambda + 2 = (\lambda - 1)^2(\lambda + 2) = 0$$

so that $\lambda = 1$, as a double root, while $\lambda = -2$ is a simple root.

The general solution is

$$y = (A_1 + A_2x)e^x + A_3e^{-2x}$$

Example 12.2

Solve $y_{iv} + y_{iii} - y_i - y = 0$ —— 12.6

Solution

The auxilliary equation is

$$\lambda^4 + \lambda^3 - \lambda - 1 = 0,$$

and it is easy to show that $\lambda = \pm 1$ are its roots. You can obtain the other roots by factorization thus:

$$(\lambda^2 - 1)(\lambda^2 + \lambda + 1) = \lambda^4 + \lambda^3 - \lambda - 1 = 0$$

Therefore the roots are

$$\lambda = \pm 1, \quad \lambda = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

or

$$\lambda = \pm 1, \lambda = -\frac{1}{2} \pm i\sqrt{\frac{3}{2}}$$

From this it is clear that the general solution is

$$y = A_1e^x + A_2e^{-x} + e^{-1/2x}(A_3 \cos \frac{\sqrt{3}}{2} + A_4 \sin \frac{\sqrt{3}}{2})$$

The non-homogeneous equation

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + 1 \ ...... + a_{n-1}y^1 + a_ny = Q(x),$$

where a_i $i = 1, 2, 3, ...$ n are constants that can be solved by using the method outlined in Lecture Eleven depending on the form of Q(x). To illustrate this consider the following examples.

Example 12.3

Solve $y_{iv} + y = xe^x$ —— 12.7

Solution

The auxilliary equation for the homogeneous equation $y_{iv} + y = xe^x$ is

$$\lambda^4 + 1 = 0,$$

and its roots are the four fourth roots of -1. To obtain these, note first that

$$e^{\pi i} = \cos \pi + i \sin \pi = -1$$

and $e^{2\pi i} = \cos 2\pi + i \sin 2\pi i = 1$

Therefore

$$-1 = e^{\pi i} = e^{\pi i}e^{2\pi i} = e^{3\pi i}$$

and on taking the fourth root of $e^{\pi i}$ and $e^{3\pi i}$ you will have that

$$e^{\pi i/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = (\frac{1+i}{\sqrt{2}})$$

$$e^{3\pi i} = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = \frac{(-1+i)}{\sqrt{2}}$$

Since complex roots occur in conjugates, the other two roots are

$$\frac{1-i}{\sqrt{2}}, \qquad \frac{-1-i}{\sqrt{2}}$$

The complimentary function is

$$y_h(x) = A_1e^{x/\sqrt{2}} \cos \frac{x}{\sqrt{2}} + A^2e^{x/\sqrt{2}} \sin \frac{x}{\sqrt{2}}$$
$$+ A_3e^{-x/\sqrt{2}} \cos \frac{x}{\sqrt{2}} + A_4e^{-x/\sqrt{2}} \sin \frac{x}{\sqrt{2}}$$

To obtain a P. I. make a trial solution of

$$y = e^x(\alpha x + \beta)$$

where α, β are to be determined. Direct differentiation four times gives

$$y_{iv} = (\alpha x + \beta)e^x + 4\alpha e^x$$

and substitution in 12.7 yields

$$(\alpha x + \beta + 4\alpha)e^x + (\alpha x + \beta)e^x = xe^x.$$

Equating correspondingly you will now obtain

$$4\alpha + 2\beta = 0, 2\alpha = 1$$

so that $\alpha = \frac{1}{2}$; $\beta = -1$. The P.I. is

$$y_p(x) = e^x(\frac{x}{2} - 1)$$

and the general solution of 12.7 is

$$y = e^{x/\sqrt{2}}(A_1 \cos\frac{x}{\sqrt{2}} + A_2 \sin\frac{x}{\sqrt{2}}) + e^{-x/\sqrt{2}}(A_3\cos\frac{x}{\sqrt{2}} + A_4 \sin\frac{x}{\sqrt{2}}) + e^x(\frac{x}{2} - 1).$$

Example12.4

Solve $y^{III} - 3y^{I} + 2y = \sin x$ —— 12.8

Solution

The reduced equation here is the equation 12.5 which was solved earlier. Therefore the complimentary function is

$$y_n(x) = (A_1 + A_2x)\bar{x}^x + A_3 e^{+2x}$$

where A_1, A_2, A_3 are arbitrary constants. To obtain a P. I. you may make a trial solution of

$$y = \alpha e^{ix}.$$

On substituting in

$$y^{III} - 3y^{I} + 2y = e^{ix}$$

you will have that

$$\alpha(2 - 4i)e^{ix} = e^{ix},$$

and so

$$\alpha = \frac{1}{2-4i}$$

The P. I. is

$$y_p(x) = 1m\left(\frac{e^{ix}}{2-4i}\right)$$

$$= 1m[(\frac{1+2i}{10}) \cos x + i \sin x)$$

$$= \frac{1}{10}(2 \cos x + \sin.x).$$

The general solution of 12.8 is

$$y = (A_1 + A_2x)e^{-x} + A_3e^{+2x} + \frac{1}{10}(2 \cos x + \sin x)$$

Post-Test

1. Solve the following equations. When initial values are given find the corresponding particular solutions.
   (a) $y^{III} - 3y^{I} + 2y = 0$, $y(0) = 1$, $y^{I}(0) = 0$, $y^{II}(0) = 0$
   (b) $y^{III} - 27y = 0$
   (c) $y_{iv} - 5y^{II} + 4y = 0$
   (d) $y^{(4)} + y = 0$, $y(0) = 1$, $y^{I}(0) = y^{II}(0) = y^{III}(0) = 0$
   (e) $y^{III} - 5y^{II} - 2y^{I} + 24y = 0$, $y(0) = 1$, $y^{I}(0) = 0$, $y^{II}(0) = -1$

2. Find a particular integral for each of the following:
   (a) $y^{III} - 5y^{II} - 2y^{I} + 24y = x$
   (b) $y_{iv} - y = \sin x$
   (c) $y^{III} - 3y^{I} + 2y = x^2 + e^{-x}$
   (d) $y^{III} - y^{I} = e^{x}(x + 1)$
   (e) $y^{III} - 3y^{II} - 3y^{I} - y = \log x$.

Difference Equations

Linear First Order Difference Equations

Introduction

You are probably familiar with the notion of 'sequence of real numbers'. For example:

... -3, -1, 1, 3, 5, ... is the sequence of odd numbers specifically, a set of numbers in a definite order of occurence is called a sequence. This may be represented by

$$...U_{-2}, U_{-1}, U_0, U_1, U_2, ...$$

Here n specifies the number u_n, so that if n is a given (prescribed) U_n may be evaluated and hence by allowing n to take values ...-2, -1, 0, 1, 2, ... the entire sequence may be determined.

For the sequence of odd numbers ... -3, -1, 1, 3, 5... $U_n = (2n + 1)$. In applications, sequences are usually required in which there is a relation between groups of consecutive terms. Such relations may take any of the forms

$$U_n - 3U_{n-1} = n \qquad n > 1$$

$$U_{n+2} - 3U_{n+1} + 2U_n = 0,$$

or more generally

$U_{n+k} + a_1U_{n+k-1} + \ldots + a_kU_n = f(n), \quad n \geq 1$ —— 13.1

A relation of the form 13.1, in which the a_i's, $i = 1, 2, \ldots k$ are constants, or in the more general case, functions of n, and f(n) is a given function of n, is called a **Difference Equation.**

In analogy with ordinary differential equation 13.1 is called a linear diference equation of order k. It is linear since the left hand side involves U_{n+j}, , $j = 0, 1, \ldots k$ linearly.

Consider the much simpler equation

$U_{n+1} = aU_n$ —— 13.2

where a is a given constant. Observe that by proceeding inductively, you will have that

$U_1 = aU_0, U_2 = aU_1 = a^2U_0$

and, in general that

$U_{n+1} = aU_n = a(aU_{n-1}) = \ldots = a^{n+1}U_0,$ —— 13.3

which is the general solution of 13.2. If you compare 13.2 and its solution 13.3 with the differential equation $y_{\prime} = ay$, which has the general solution. $y = ce^{ax}$, you will see readily that $n + 1$, a and u_0 correspond respectively to x, e^a and c.

Consider next the more general equation

$U_{n+1} = a_nU_n$ —— 13.4

Proceeding as in the above, you have that $U_1 = a_0U_0$, $U_2 = a_1U_1 = a_1a_0U_0$, ... and in general that

$U_{n+1} = a_na_{n-1} \ldots a_1a_0U_0 = (\prod_{k=0}^{n} a_k)U_0$ —— 13.5

The comparable differential equation in this case is $y' = a(x)y$, with general solution $y = ce^{\int a(x)dx}$, so that $\prod_{k=0}^{n} a_k$ corresponds to $e^{\int a(x)dx}$

Suppose now that the problem is to solve

$U_{n+1} = U_n + f_n$ —— 13.6

The iterative procedure will give

$U_1 = U_0 + f_0, U_2 = U_1 + f_1 = U_0 + f_0 + f_1, \ldots,$ so that

$U_{n+1} = U_0 + \sum_{k=0}^{n} f_k$ —— 13.7

since $y_{\prime} = y + f(x)$ has the general solution $y = ce^x + e^x \int f(x)e^{-x}dx$, the correspondence is clear if you note that $\prod_{k=0}^{n} 1 = 1$.

Finally consider the most general first order linear equation

$$U_{n+1} = a_nU_n + f_n \qquad \text{—— 13.8}$$

Proceeding inductively, you will have that

$U_1 = a_0y_0 + f_0,\ U_2 = a_1U_1 + f^1 = a_1a_0y_0 + a_1f_0 + f_1$

and, in general

$$U_{n+1} = (a_na_{n-1}......a_1a_0)\,U_0 + (a_n\,...a_1)f_0 + (a_n....a_{n2})f_1$$

$$+........+a_nf_{n-1}+f_n \qquad \text{——13.9}$$

$$= \left(\prod_{k=0}^{n} a_k\right)U_0 + \sum_{k=0}^{n}\left(\prod_{k=0}^{n} a_j\right)f_k.$$

You may wish to compare this equation with

$$y = e^{-\int a(x)dx}\left[\int f(x)e^{\int a(x)dx}dx + c\right] \qquad \text{—— 13.10}$$

which is the general solution of $y' + a(x)y = f(x)$. Indeed 13.9 is the discrete analog of 13.10. Observe in 13.9 that the empty product is defined by $\prod_{j=n+1}^{n} a_j = 1$

Example 13.1

Solve

(i) $y_{n+1} - y_n = 2^{-n}$

(ii) $y_{n+1} = \dfrac{n+5}{n+3}\,y_n$

Hint

(i) is of the form 13.6, and using 13.7, you will see that its solution is

$$y_{n+1} = y_0 + \sum_{k=0}^{n} 2^{-k}$$

Also (ii) is of the form 13 4 and by 13.5 its solution is

$$y_{n+1} = \prod_{k=0}^{n} \frac{(k+5)}{(k+3)}\,y_0 = \frac{1}{12}\,(n+4)(n+5)\,y_0$$

Example 13.2

An amoeba population has an initial size of 1000. It is observed, on every ten amoebas reproduce by cell division every hour. Find the approximate size of the amoeba population in 30 hours.

Solution

Let y_n denote the number of amoebas after n hours. Then the population growth over the next hour is

$y_{n+1} - y_n = \frac{1}{10} y_n;$

that is $y_{n+1} = (1.1)y_n$.

Using 13.3, you will have that

$y_{n+1} = (1.1)^{n+1} y_0, \; y_0 = 1000$

In 30 hours the population size is

$$y_{30} = (1.1)^{30}\, 10^3 \approx 17449.$$

Example 13.3

Suppose in the preceding example that a leak from another container is introducing 20 additional amoeba into the population per hour. What is the population size in 30 hours.

The governing equation becomes

$y_{n+1} - y_n = \frac{1}{10} y_n + 20$

or $y_{n+1} = (1.1)y_n + 20,$

which, by 13.9, has the solution

$$\begin{aligned} y_{n+1} &= 1000(1.1)^{n+1} + \sum_{k=0}^{n} (1.1)^{n-k}(20) \\ &= 1000(1.1)^{n+1} + 20\left(\frac{(1.1)^{n+1} - 1}{1.1 - 1}\right) \\ &\approx 17449 + 3289.8 \\ &= 20738.8 \end{aligned}$$

Post-Test

In problems 1 to 5 find the general solution of each difference equation and a particular solution when an initial condition is specified.

1. $y_{n+1} + y_n = 3^{-n}$
2. $y_{n+1} = \frac{2n+3}{2n+1} y_n$
3. $(n+1)y_{n+1} = (n+2)y_n, \; y_0 = 1$
4. $y^{n+1} - ny_n = n! \;\; y_0 = 5$
5. $y_{n+1} - e^{2n} y_n = e^{n^2}$

6. In a population model it is assumed that the probability p_n that a couple produces exactly n offsprings satisfies the equation $p_n = 0.7p_{n-1}$. Find p_n in terms of p_0 and determine p_0 from the fact that $p_0 + p_1 + p_2 + \ldots + = 1$. An alternative model is $p_n = (\frac{1}{n})p_{n-1}$ For this model find p_n in terms of p_0 and prove that $p_0 = {}^1/e$.

Properties of Solutions

Introduction

It is convenient to start with the homogeneous second order linear equation

$$y_{n+2} + a_n y_{n+1} + b_n y_n = 0 \qquad \text{—— 14.1}$$

A solution of 14.1 is a sequence of real numbers $y_0, y_1, y_2 ... y_n, ...$ that satisfies equation 14.1 for all integers $n \geq 1$. For convenience such a solution is denoted simply by the term y_n. A linear combination of the solutions x_n, y_n of 14.1 is simply the sequence $Ax_n + By_n$, where A, B are real constants. Two solutions x_n, y_n of 14.1 are linearly independent if $Ax_n + By_n = 0$ implies that $A = B = 0$. This is equivalent to saying that there is no constant c such that $x_n = cy_n$ for all integers n. It is easy to show that any linear combination of solutions of 14.1 is also a solution of 14.1. To see this let $Z_n = Ax_n + By_n$, where x_n, y_n are solutions of 14.1.

Then

$$\begin{aligned} & Z_{n+2} + a_n Z_{n+1} + b_n Z_n \\ = \; & Ax_{n+2} + By_{n+2} + a_n(Ax_{n+1} + By_{n+1}) + b_n(Ax_n + By_n) \\ = \; & A(x_{n+2} + a_n x_{n+1} + b_n x_n) + B(y_{n+2} + a_n y_{n+1} + b_n y_n) = 0, \end{aligned}$$

which shows that the linear combination of x_n, y_n is also a solution of 14.1. Let x_n, y_n be two solutions of 14.1 and let

$$c_n(x,y) = x_n y_{n+1} - x_{n+1} y_n$$

$c_n(x,y)$ is called the **casoration** of the difference equation 14.1, it is the analog of the Wronskian of a second order linear ordinary differential equation. It turns out that:

I. If $b_n \neq 0$ for all integers $n \geq 0$ then equation 14.1 has two linearly independent solutions.

II. If $b_n \neq 0$ for all integer $n \geq 0$ and x_n, y_n are solutions of 14.1, then x_n, y_n are linearly independent if and only if $c_n(x,y) \neq 0$ for some integer n.

Homogeneous Equation with Constant Co-efficients

Consider the linear homogeneous equation with constant co- efficients

$$y_{n+2} + a y_{n+1} + b y_n = 0, \ b \neq 0 \qquad \text{—— 14.2}$$

where a, b are constants, which do not depend on n. In the case of first order equation $y_{n+1} = a y_n$ you saw in Lecture 13 that its general solution is $y_n = a^n y_0$. It is therefore plausible to guess that 14.2 has solutions of the form $y_n = \lambda^n$ for some (real or complex) number λ. Indeed, on making $y_n = \lambda^n$ a trial solution of 14.2, you will have that λ must satisfy the equation

$$\lambda^{n+2} + a\lambda^{n+1} + b\lambda^n = 0$$

Since this equation must be true for all $n \geq 0$ if $y_n = \lambda^n$ is to be a solution of 14.2, then in particular for $n = 0$, the equation

$$\lambda^2 + a\lambda + b = 0 \qquad \text{—— 14.3}$$

must hold. This is the **auxilliary equation** for the difference equation 14.2 and is essentially the same as the auxilliary equation derived for the second order differential in Lecture eight. The roots of 14.3 are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

and as in Lectures 8 - 12 the following three cases can be distinguished:

Case I:

$a^2 - 4b > 0.$

The roots λ_1, λ_2 of 14.3 are real and distinct and the general solution of 14.2 is

$$y_n = c_1 \lambda_1^n + c_2 \lambda_2^n \qquad \text{—— 14.4}$$

where c_1, c_2 are arbitrary constants.

Case II

$a^2 - 4b = 0.$

Here the two roots of 14.3 are real and equal, and a solution of 14.2 is $y_n = \lambda^n$ where $\lambda = -\frac{a}{2}$. If you recall a similar situation for second order differential equation in lecture eight, you will see that a second linearly independent solution is $y_n = n\lambda^n$, with $\lambda = -\frac{a}{2}$. Indeed on substituting $y_n = n\lambda^n$ into the left hand side of 14.2 you will have that

$$(n + 2)\lambda^{n+2} + a(n + 1)\lambda^{n+1} + b_n\lambda^n$$
$$= \lambda^n[n\{\lambda^2 + a\lambda + b\} + (2\lambda^2 + a\lambda)] = 0$$

by 14.3 and the fact that $\lambda = -\frac{a}{2}$ Thus the general solution of 14.2 is

$$y_n = (c_1 + c_2 n)\lambda^n, \quad \lambda = -\frac{a}{2} \qquad \text{—— 14.5}$$

where c_1, c_2 are arbitrary constants.

Case III

$a^2 - 4b < 0.$

The roots of 14.3 are

$$\lambda_1 = \alpha + i\beta\, \lambda = \alpha - i\beta$$

where $\alpha = -\frac{a}{2}$, $\beta = \frac{4b - a^2}{2}$ and $i^2 = -1$.

In polar form,

$$\lambda_1 = re^{i\theta}, \quad \lambda_2 = re^{-i\theta}$$

where $r = \alpha^2 + \beta^2, \theta$ and $= \tan^{-1}(\beta/\alpha)$

$0 < \theta < \pi$. Since the solutions of 14.2 are of the form λ^n, the two linearly independent solutions are

$$y_n = (re^{i\theta})^n = r^n e^{in\theta}, \quad x_n = r^n e^{-in\theta}.$$

Since 14.2 is linear you may verify that both $(x_n + y_n)$ and $(x_n - y_n)$ are solutions of 14.2, and as in Lectures 8-12 these solutions can be written in the form

$$x_n^* = r^n \cos n\theta, \quad y_n^* = r^n \sin n\theta,$$

Verify that these solutions are linearly independent. The general solution of 14.2 is

$$y_n = c_1 r^n \cos n\theta + c_2 r^n \sin n\theta \qquad \text{—— 14.6}$$

where $r = \alpha^2 + \beta^2$, $\theta = \tan^{-1}(\beta/\alpha)$, $0 < \theta < \pi$, $\alpha = -\frac{a}{2}$, $\beta = 4b - a^2/2$ and c_1, c_2 are arbitrary constants.

Example 14.1

Solve the equation:

$$y_{n+2} - y_{n+1} - 6y_n = 0.$$

Solution

The auxilliary equation is $\lambda^2 - \lambda - 6 = 0$ with roots $\lambda_1 = 3, \lambda_2 = -2$. Therefore, the general solution is

$$y_n = c_1 3^n + c_2(-2)^n \qquad \text{—— 14.7}$$

If the initial conditions $y_0 = 1$, $y_1 = 8$ are specified, you will obtain from 14.7 the system of equations

$$c_1 + c_2 = 1$$

$$3c_1 - 2c_2 = 8$$

from which to determine c_1, c_2. Solving simultaneously you have $c_1 = 2, c_2 = -1$, and the particular solution is

$$y_n = 2.3^n - (-2)^n.$$

Example 14.2

Solve the equation

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

with initial conditions $y_0 = 1$, y_1

Solution

The auxilliary equation is $\lambda^2 - 4\lambda + 4 = 0$ with the double root $\lambda = 2$. The general solution, therefore is

$$y_n = 2^n(c_1 + nc_2)$$

Using the initial conditions $y_0 = 1$, $y_1 = -2$, you will find readily that $c_1 = 1$, $c_2 = -2$ and the particular solution is

$$y_n = 2^n(1 - 2n).$$

Example 14.3

Find the general solution of $y_{n+2} + 8y_n = 0$

Solution

The characteristic equation $\lambda + 8 = 0$ with roots $\lambda = 2i$. Here $\alpha = 0$, $\beta = 2$ and $r = 2$, $\theta = \frac{\pi}{2}$. The general solution is

$$y_n = 2^n(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2}).$$

If $y_0 = 0$, $y_1 = 2000$ are specified, you will readily obtain $c_1 = 0$, $c_2 = 1000$ and

$$y_n = 1000.2^n \sin \frac{n\pi}{2}.$$

as the particular solution.

Post-Test

In each of the problems 1-8 solve the given difference equation, and when initial conditions are specified find the unique solution that satisfies them.

1. $y_{n+2} - 3y_{n+1} - 4y_n = 0$
2. $y_{n+2} + 7y_{n+1} + 12y_n = 0$ $y_0 = 0$, $y^1 = 1$
3. $3y_{n+2} - 2y_{n+1} - y_n = 0$ $y_0 = 0$, $y_1 = 3$
4. $9y_{n+2} + 6y_{n+1} + y_n = 0$
5. $y_{n+2} - \sqrt{2}y_{n+1} + y_n = 0$
6. $y_{n+2} + y_n = 0$
7. $y_{n+2} + y_{n+1} + y_n = 0$
8. $y_{n+2} - 2y_{n+1} + 4y_n = 0$, $y_0 = 0$, $y_1 = 1$
9. The Fibonacci numbers are a sequence of numbers such that each one is the sum of its two predecessors. The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, ...

   (a) Formulate an initial value difference equation that will generate the Fibonacci numbers.

   (b) Find the solution to this equation

   (c) Show that the ratio of successive Fibonacci numbers tends to $(1 + \sqrt{5})/2$ as $n \to \infty$

10. Consider the integrals

$$In(\theta) = \int_0^{\pi} \frac{\pi \cos nx - \cos n\theta \, dx}{\cos x - \cos \theta}$$

(a) Show that $I_{n+2}(\theta) = 2 \cos\theta \, I_{n+1}(\theta) \; I_n(\theta)$

(b) Solve this equation and obtain an expression for $I_n(0)$.

(Hint: First determine $I_0(0)$ and $I_1(0)$).

LECTURE FIFTEEN

Non-homogeneous Equation with Constant Co-efficient

Introduction

Consider the linear non-homogeneous equation with constant co- efficients:

$$y_{n+2} + ay_{n+1} + by_n = f_n \qquad \text{— 15.1}$$

where a, b are constants which do not depend on n. You saw in Lecture 9 how the method of undetermined co-efficients is used to find particular solutions of non-homogeneous linear second order ordinary differential equations with constant co- efficients. The same technique applies to equation 15.1 when f_n is one of the following forms

1. $A_0 r^n = z_n$
2. $z_n = A_1 \sin cn + A_2 \cos cn$
3. $z_n = A_0 + A_1 n + ... + A_k n^k$

or any combination of these terms. To solve 15.1, with f_n in one of above forms, you will proceed as follows:

Find the independent solutions x_n, y_n of the homogeneous equation

$$y_{n+2} + ay_{n+1} + by_n = 0 \qquad \text{— 15.2}$$

Next write the expression z_n in the same form as f_n with undetermined co-efficients A_j.

(i) If no part of z_n is in the general solution of the homogeneous equation 15.2, substitute z_n for y_n in the equation 15.1, and solve for the co-efficients A_j

(ii) Otherwise, multiply z_n by the smalllest integral power of n such that no part of the product belongs to the general solution of the homogeneous equation 15.2, and proceed as with z_n in step (i). As in lecture nine this method of solution is best illustrated with examples.

Case I

Consider $y_{n+2} - 5y_{n+1} + 6y_n = 5^{n+1}$ —— 15.3

The general solution of the homogeneous equation is

$y_{n,g} = A(2)^n + B(3)^n$

By (i), since 5^{n+1} does not appear in $y_{n,g}$ make a trial solution of $z_n = A_0 5^{n+1}$, where A_0 is to be determined. Substituting z_n for y_n in 15.3 you will have

$A_0(5^{n+3} - 5.5^{n+2} + 6.5^{n+1}) = 5^{n+1}$

that is $A_0 = \frac{1}{6}$

Thus $z_n = \frac{1}{6}5^{n+1}$ is a particular solution and the general solution of 15.3 is

$y_n = A_0(2)^n + B3^n + \frac{5^{n+1}}{6}$.

As indicated in (ii) this method may fail if the right hand side of 15.1 is a part of the general solution of the homogeneous equation 15.2. To illustrate consider

$y_{n+2} - 5y_{n+1} + 6y_n = 2^n$ —— 15.4

An attempt to make $z_n = A_0 2^n$ a trial solution will lead to

$A_0(2^2 - 5.2 + 6)2^n = 2^n$,

showing that A_0 is indeterminate. As indicated in (ii), the way out is to multiply z_n by n so that $z_n = A_0 n.2^n$ is not a solution of the homogeneous equation $y_{n+2} - 5y_{n+1} + 6y_n = 0$.

Indeed, on substituting $z_n = A_0 n2^n$ for y_n in 15.4 you will have

$A_0[(n+2)2^{n+2} - 5(n+1)2^{n+1} + 6n2^n] = 2^n$

that is $A_0[(n+2)2^2 - 5(n+1)2 + 6n] = 1$

Thus $A_0 = -\frac{1}{2}$ and the particular solution is

$Z_n = -\frac{1}{2} n .2_n$

The general solution s

$$y_n = (A - \frac{n}{2})2^n + B3^n$$

Case 2

$$y_{n+2} + 8y_n = \sin\frac{n\pi}{4} \qquad \text{—— 15.5}$$

The general solution of corresponding homogeneous equation you already obtain in Example 14.1 as

$$y_{n,g} = 2^n(c_1 \cos n\frac{\pi}{2} + c_2 \sin n\frac{\pi}{2})$$

It is convenient to work in polar co-ordinates and make a trial solution of

$$Z_n = \quad e^{\frac{in\pi}{4}},$$

where A_0 is to be determined.

Substitution in 15.5 gives

$$A_0[e^{i(n-2)\frac{\pi}{4}} + 8e^{in\frac{\pi}{4}}] = e^{in\frac{\pi}{4}}$$

so that

$$A_0 = \frac{1}{8+i}$$

Therefore the particular integral is

$$y_{n,p} = \text{Im}\left[\frac{1}{8-i} e^{\frac{in\pi}{4}}\right]$$

$$= \text{Im}\left[\frac{8+i}{65}(\cos n\frac{\pi}{4} + i\sin n\frac{\pi}{4})\right]$$

$$= \frac{1}{65}(8 \sin n\frac{\pi}{4} - \cos n\frac{\pi}{4})$$

and the general solution is

$$y_n = 2^n(c_1 \cos\frac{n\pi}{2} + c_2 \sin\frac{n\pi}{2}) + \frac{1}{65}(8 \sin\frac{n\pi}{4} - \cos\frac{n\pi}{4})$$

Case 3

Consider $y_{n+2} - 5y_{n+1} + 6y_n = n$ —— 15.6

Here make a trial solution of $z_n = A_0n + A_1$

where A_0, A_1 are to be determined substituting in 15.6 will give

$$A_0(n+2) + A_1 - 5A_0(n+1) - 5A_1 + 6A_0n + 6A_1 = n$$

and on equating corresponding co-efficient of powers of n, you have that

$$A_0 - 5A_0 + 6A_0 = 1,$$

$$2A_0 + A_1 - 5A_0 - 5A_1 + 6A_1 = 0$$

Therefore $A_0 = 1/2$ and $A_1 = \frac{3}{4}$, and a particular solution is

$$z_n = \frac{2n + 3}{4}$$

The general solution of 15.6 is

$$y_n = A2^n + B3^n + \frac{2n + 3}{4}$$

Post-Test

Solve each of the following equations:

1. $y_{n+2} - y_{n+1} - 12y_n = 2^n$
2. $y_{n+2} - 4y_{n+1} + 4y_n = 2^n$
3. $y_{n+2} + y_{n+1} - 6y_n = e^{n+1} + 3e^n$
4. $y_{n+2} + 5y_n = \sin \frac{n\pi}{2}$
5. $y_{n+2} + y_n = 3^n \cos \frac{n\pi}{4}$
6. $y_{n+2} - 3y_{n+1} + 2y_n = n2^n + 2^{-n}$
7. $y_{n+3} - 2y_{n+2} + 3y_n = \cos \frac{n\pi}{2}$
8. $y_{n+3} + y_{n+2} - 2y_n = n + n^2 + n^3$
9. $y_{n+2} + y_{n+1} + \frac{1}{4}y_n = 2^n + 3^n$
10. $y_{n+2} + 8y_n = \sin 2n.$